12.E: Applications of Functions of Several Variables (Exercises)

12.1: Introduction to Multivariable Functions

Terms and Concepts

1. Give two examples (other than those given in the text) of "real world" functions that require more than one input.

2. The graph of a function of two variables is a _______.

3. Most people are familiar with the concept of level curves in the context of _____ maps.

4. T/F: Along a level curve, the output of a function does not change.

5. The analogue of a level curve for functions of three variables is a level _______.

6. What does it mean when level curves are close together? Far apart?

Problems

In Exercises 7-14, give the domain and range of the multivariable function.

7. \((f(x,y) = x^2+y^2+2)\)
8. \( f(x,y) = x+2y \)
9. \( f(x,y) = x-2y \)
10. \( f(x,y) = \frac{1}{x+2y} \)
11. \( f(x,y) = \frac{1}{x^2+y^2+1} \)
12. \( f(x,y) = \sin x \cos y \)
13. \( f(x,y) = \sqrt{9-x^2-y^2} \)
14. \( f(x,y) = \frac{1}{\sqrt{x^2+y^2-9}} \)

In Exercises 15-22, describe in words and sketch the level curves for the function and given \( c \) values.

15. \( f(x,y)=3x-2y; \, c=-2,0,2 \)
16. \( f(x,y)=x^2-y^2; \, c=-1,0,1 \)
17. \( f(x,y)=x\cdot y^2; \, c=-2,0,2 \)
18. \( f(x,y)=\frac{1-x^2-y^2}{2y-2x}; \, c=-2,0,2 \)
19. \( f(x,y)=\frac{1-x^2-y^2}{x^2+y^2+1}; \, c=-1,0,1 \)
20. \( f(x,y)=y-x^3-3; \, c=-3,-1,1,3 \)
21. \( f(x,y)=\sqrt{x^2+4y^2}; \, c=1,2,3,4 \)
22. \( f(x,y)=x^2+4y^2; \, c=1,2,3,4 \)

In Exercises 23-26, give the domain and range of the functions of three variables.

23. \( f(x,y,z)=\frac{x}{x+2y-4z} \)
24. \( f(x,y,z)=\frac{1}{1-x^2-y^2-z^2} \)
25. \( f(x,y,z)=\sqrt{z-x^2+y^2} \)
26. \( f(x,y,z) = z^2 \sin x \cos y \)

In Exercises 27-30, describe the level surfaces of the given functions of three variables.

27. \( f(x,y,z) = x^2+y^2+z^2 \)
28. \(f(x,y,z) = z-x^2+y^2\)

29. \(f(x,y,z) = \frac{x^2+y^2}{z}\)

30. \(f(x,y,z) = \frac{z}{x-y}\)

31. Compare the level curves of Exercises 21 and 22. How are they similar, and how are they different? Each surface is a quadric surface; describe how the level curves are consistent with what we know about each surface.

### 12.2: Limits and Continuity of Multivariable Functions

#### Terms and Concepts

1. Describe in your own words the difference between boundary and interior point of a set.

2. Use your own words to describe (informally) what \(\lim_{(x,y)\to (1,2)}f(x,y)=17\) means.

3. Give an example of a closed, bounded set.

4. Give an example of a closed, unbounded set.

5. Give an example of a open, bounded set.

6. Give an example of a open, unbounded set.

#### Problems

In Exercises 7-10, a set \(S\) is given.

(a) Give one boundary point and one interior point, when possible, of \(S\).

(b) State whether \(S\) is open, closed, or neither.

(c) State whether \(S\) is bounded or unbounded.

7. \((S=\{ (x,y) \mid (x-1)^2+4(y-3)^2 \le 1 \})\)

8. \((S=\{ (x,y) \mid y \text{ or } x^2 \le 1 \})\)

9. \((S=\{ (x,y) \mid x^2+y^2=1 \})\)

10. \((S=\{ (x,y) \mid y > \sin x \})\)

In Exercises 11-14:

(a) Find the domain \(D\) of the given function.
(b) State whether \(D\) is open or closed set.
(c) State whether \(D\) is bounded or unbounded.

11. \(f(x,y) = \sqrt{9-x^2-y^2}\)

12. \(f(x,y) = \sqrt{y-x^2}\)

13. \(f(x,y) = \frac{1}{\sqrt{y-x^2}}\)

14. \(f(x,y) = \frac{x^2-y^2}{x^2+y^2}\)

In Exercises 15-20, a limit is given. Evaluate the limit along the paths given, then state why these results show the given limit does not exist.

15. \(\lim\limits_{{(x,y) \to (0,0)}} \frac{x^2-y^2}{x^2+y^2}\)
   (a) Along the path \((y=0)\).
   (b) Along the path \((x=0)\).

16. \(\lim\limits_{{(x,y) \to (0,0)}} \frac{x+y}{x-y}\)
   (a) Along the path \((y=mx)\).
   (b) Along the path \((x=0)\).

17. \(\lim\limits_{{(x,y) \to (0,0)}} \frac{xy-y^2}{y^2+x}\)
   (a) Along the path \((y=mx)\).
   (b) Along the path \((x=0)\).

18. \(\lim\limits_{{(x,y) \to (0,0)}} \frac{\sin (x^2)}{y}\)
   (a) Along the path \((y=mx)\).
   (b) Along the path \((y=x^2)\).

19. \(\lim\limits_{{(x,y) \to (1,2)}} \frac{x+y-3}{x^2-1}\)
   (a) Along the path \((y=2)\).
   (b) Along the path \((y=x+1)\).

20. \(\lim\limits_{{(x,y) \to (\pi, \pi/2)}} \frac{\sin x}{\cos y}\)
   (a) Along the path \((x=\pi)\).
   (b) Along the path \((y=x-\pi/2)\).

12.3: Partial Derivatives

Terms and Concepts

1. What is the difference between a constant and a coefficient?
2. Given a function \( z = f(x,y) \), explain in your own words how to compute \( f_x \).

3. In the mixed partial fraction \( f_{xy} \), which is computed first, \( f_x \) or \( f_y \)?

4. In the mixed partial fraction \( \frac{\partial^2 f}{\partial x \partial y} \), which is computed first, \( f_x \) or \( f_y \).

**Problems**

In Exercises 5-8, evaluate \( f_x(x,y) \) and \( f_y(x,y) \) at the indicated point.

5. \( f(x,y) = x^2y - x + 2y + 3 \) at \((1,2)\)

6. \( f(x,y) = x^3 + 3x^2y - 2y^2 - 6y \) at \((-1,3)\)

7. \( f(x,y) = \sin y \cos x \) at \((\pi/3, \pi/3)\)

8. \( f(x,y) = \ln (xy) \) at \((-2,-3)\)

In Exercises 9-26, find \( f_x, f_y, f_{xx}, f_{yy}, f_{xy} \) and \( f_{yx} \).

9. \( f(x,y) = x^2y + 3x^2 + 4y - 5 \)

10. \( f(x,y) = y^3 + 3xy^2 + 3x^2y + x^3 \)

11. \( f(x,y) = \frac{x}{y} \)

12. \( f(x,y) = \frac{4}{xy} \)

13. \( f(x,y) = e^{x^2+y^2} \)

14. \( f(x,y) = e^{x+2y} \)

15. \( f(x,y) = \sin x \cos y \)

16. \( f(x,y) = (x+y)^3 \)

17. \( f(x,y) = \cos (5xy^3) \)

18. \( f(x,y) = \sin (5x^2 + 2y^3) \)

19. \( f(x,y) = \sqrt{4xy^2 + 1} \)

20. \( f(x,y) = (2x+5y)\sqrt{y} \)

21. \( f(x,y) = \frac{1}{x^2+y^2+1} \)
22. \( f(x,y) = 5x - 17y \)

23. \( f(x,y) = 3x^2 + 1 \)

24. \( f(x,y) = \ln(x^2 + y) \)

25. \( f(x,y) = \frac{\ln x}{4y} \)

26. \( f(x,y) = 5e^x \sin y + 9 \)

In Exercises 27-30, form a function \( z = f(x,y) \) such that \( f_x \) and \( f_y \) match those given.

27. \( f_x = \sin y + 1, \quad f_y = x \cos y \)

28. \( f_x = x + y, \quad f_y = x + y \)

29. \( f_x = 6xy - 4y^2, \quad f_y = 3x^2 - 8xy + 2 \)

30. \( f_x = \frac{2x}{x^2 + y^2}, \quad f_y = \frac{2y}{x^2 + y^2} \)

In Exercises 31-34, find \( f_x, f_y, f_z, f_{yz} \) and \( f_{zy} \).

31. \( f(x,y,z) = x^2 e^{2y - 3z} \)

32. \( f(x,y,z) = x^3 y^2 + x^3 z + y^2 z \)

33. \( f(x,y,z) = \frac{3x}{7y^2 z} \)

34. \( f(x,y,z) = \ln(xyz) \)

12.4: Differentiability and the Total Differential

Terms and Concepts

1. T/F: If \( f(x,y) \) is differentiable on \( S \), the \( f \) is continuous on \( S \).

2. T/F: If \( f_x \) and \( f_y \) are continuous on \( S \), then \( f \) is differentiable on \( S \).

3. T/F: If \( z = f(x,y) \) is differentiable, then the change in \( z \) over small changes \( \Delta x \) and \( \Delta y \) in \( x \) and \( y \) is approximately \( \Delta z \).

4. Finish the sentence: "The new \( z \)-value is approximately the old \( z \)-value plus the approximate \( \ldots \)."
Problems

In Exercises 5-8, find the total differential $dz$.

5. $z = x\sin y + x^2$
6. $z = (2x^2 + 3y)^2$
7. $z = 5x - 7y$
8. $z = xe^{x+y}$

In Exercises 9-12, a function $z = f(x, y)$ is given. Give the indicated approximation using the total differential.

9. $f(x, y) = \sqrt{x^2 + y^2}$. Approximate $f(2, 95, 7.1)$ knowing $f(3, 7) = 4$.
10. $f(x, y) = \sin x \cos y$. Approximate $f(0.1, -0.1)$ knowing $f(0, 0) = 0$.
11. $f(x, y) = x^2y - xy^2$. Approximate $f(2.04, 3.06)$ knowing $f(2, 3) = -6$.
12. $f(x, y) = \ln (x-y)$. Approximate $f(5.1, 3.98)$ knowing $f(5, 4) = 0$.

Exercises 13-16 ask a variety of questions dealing with approximating error and sensitivity analysis.

13. A cylindrical storage tank is to be 2ft tall with a radius of 1ft. Is the volume of the tank more sensitive to changes in the radius or the height?

14. Projectile Motion: The $x$-value of an object moving under the principles of projectile motion is $x(\theta, v_0, t) = (v_0 \cos \theta)t$. A particular projectile is fired with an initial velocity of $v_0 = 250$ ft/s and an angle of elevation of $\theta = 60^\circ$. It travels a distance of 375ft in 3 seconds. Is the projectile more sensitive to errors in initial speed or angle of elevations?

15. The length $l$ of a long wall is to be approximated. The angle $\theta$, as shown in the diagram (not to scale), is measured to be $85^\circ$, and the distance $x$ is measured to be $30'$. Assume that the triangle formed is a right triangle. Is the measurement of the length of $l$ more sensitive to errors in the measurement of $x$ or $\theta$?

16. It is "common sense" that it is far better to measure a long distance with a long measuring tape rather than a short one. A measured distance $D$ can be viewed as the product of the length $l$ of a measuring tape times the number $n$ of times it was used. For instance, using a 3' tape 10 times gives a length of 30'. To measure the same distance with a 12' tape, we would use the tape 2.5 times. (i.e., $30 = 12 \times 2.5$.) Thus $D = nl$.

Suppose each time a measurement is taken with the tape of the recorded distance within $\frac{1}{16''}$ of the actual distance, (i.e., $dl = 0.005''$). Using differentials, show why common sense proves correct in that it is better to use a long tape...
to measure long distances.

In Exercises 17-18, find the total differential \( dw \).

17. \( w=x^2yz^3 \)

18. \( w=e^{x\sin y \ln z} \)

In Exercises 19-22, use the information provided and the total differential to make the given approximation.

19. \( f(3,1)=7, f_x(3,1)=9, f_y(3,1)=-2 \) Approximate \( f(3.05, 0.9) \).

20. \( f(-4,2)=13, f_x(-4,2)=2.6, f_y(-4,2)=5.1 \) Approximate \( f(-4.12,2.07) \).

21. \( f(2,4,5)=-1, f_x(2,4,5)=2, f_y(2,4,5)=-3, f_z(2,4,5)=3.7 \) Approximate \( f(2.5, 4.1,4.8) \).

22. \( f(3,3,3)=5, f_x(3,3,3)=2, f_y(3,3,3)=0, f_z(3,3,3)=-2 \) Approximate \( f(3.1,3.1,3.1) \).

12.5: The Multivariable Chain Rule

Terms and Concepts

1. Let a level curve of \( z=f(x,y) \) be described by \( x=g(t), \ y=h(t) \). Explain why \( \frac{dz}{dt}=0 \).

2. Fill in the blank: The single variable Chain Rule states \( \frac{d}{dx} \left( f\left( g(x) \right) \right) = f' \left( g(x) \right) \cdot \)_________.

3. Fill in the blank: The Multivariable Chain Rule states \( \frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \)_________ + \__________ \cdot \frac{dy}{dt} \).

4. If \( z=f(x,y) \), where \( x=g(t) \) and \( y=h(t) \), we can substitute and write \( z \) as an explicit function of \( t \).

T/F: Using the Multivariable Chain Rule to find \( \frac{dz}{dt} \) is sometimes easier than first substituting and then taking the derivative.

5. T/F: The Multivariable Chain Rule is only useful when all the related functions are known explicitly.

6. The Multivariable Chain Rule allows us to compute implicit derivatives easily by just computing two ______ derivatives.

Problems

In Exercises 7-12, functions \( z=f(x,y), x=g(t) \) and \( y=h(t) \) are given.

(a) Use the Multivariable Chain Rule to compute \( \frac{dz}{dt} \).
(b) Evaluate $\frac{dz}{dt}$ at the indicated $t$-value.

7. $(z=3x+4y, \quad x=t^2, \quad y=2t; \quad t=1)$

8. $(z=x^2-y^2, \quad x=t, \quad y=t^2-1; \quad t=1)$

9. $(z=5x+2y, \quad x=2\cos t+1, \quad y=\sin t-3; \quad t=\pi/4)$

10. $(z=\frac{x}{y^2+1}, \quad x=\cos t, \quad y=\sin t; \quad t=\pi/2)$

11. $(z=x^2+2y^2, \quad x=\sin t, \quad y=3\sin t; \quad t=\pi/4)$

12. $(z=\cos x \sin y, \quad x=\pi t, \quad y=2\pi t+\pi/2; \quad t=3)$

In Exercises 13-18, functions $z=f(x,y), x=g(t)$ and $y=h(t)$ are given. Find the values of $t$ where $\frac{dz}{dt}=0$.

Note: these are the same surfaces/curves as found in Exercises 7-12.

13. $(z=3x+4y, \quad x=t^2, \quad y=2t)$

14. $(z=x^2-y^2, \quad x=t, \quad y=t^2-1)$

15. $(z=5x+2y, \quad x=2\cos t+1, \quad y=\sin t-3)$

16. $(z=\frac{x}{y^2+1}, \quad x=\cos t, \quad y=\sin t)$

17. $(z=x^2+2y^2, \quad x=\sin t, \quad y=3\sin t)$

18. $(z=\cos x \sin y, \quad x=\pi t, \quad y=2\pi t+\pi/2)$

In Exercises 19-22, functions $z=f(x,y), x=g(s,t)$ and $y=h(s,t)$ are given.

(a) Use the Multivariable Chain Rule to compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

(b) Evaluate $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at the indicated $s$ and $t$ values.

19. $(z=x^2y, \quad x=s-t, \quad y=2s+4t; \quad s=1, t=0)$

20. $(z=\cos \sqrt{\left(\frac{\pi}{2}\right)} \cdot \frac{\pi}{2}, \quad x=st^2, \quad y=s^2t; \quad s=1, t=1)$

21. $(z=x^2+y^2, \quad x=s\cos t, \quad y=s\sin t; \quad s=2, t=\pi/4)$

22. $(z=e^{-(x^2+y^2)}, \quad x=t, \quad y=st^2; \quad s=1, t=1)$

In Exercises 23-26, find $\frac{dy}{dx}$ using Implicit Differentiation and Theorem 109.

23. $(x^2 \tan y=50)$

24. $(3x^2+2y^3)^4=2)$
25. \(\frac{x^2+y}{x+y^2}=17\)

26. \(\ln (x^2+xy +y^2)=1\)

In Exercises 27-30, find \(\frac{dz}{dt}\), or \(\frac{\partial z}{\partial s}\) and \(\frac{\partial z}{\partial t}\), using the supplied information.

27. \(\frac{\partial z}{\partial x}=2, \quad \frac{\partial z}{\partial y}=1, \quad \frac{dx}{dt}=4, \quad \frac{dy}{dt}=-5\)

28. \(\frac{\partial z}{\partial x}=1, \quad \frac{\partial z}{\partial y}=-3, \quad \frac{dx}{dt}=6, \quad \frac{dy}{dt}=2\)

29. \(\frac{\partial z}{\partial x}=-4, \quad \frac{\partial z}{\partial y}=9, \quad \frac{\partial x}{\partial s}=5, \quad \frac{\partial y}{\partial s}=-2, \quad \frac{\partial x}{\partial t}=7, \quad \frac{\partial y}{\partial t}=6\)

30. \(\frac{\partial z}{\partial x}=2, \quad \frac{\partial z}{\partial y}=1, \quad \frac{\partial x}{\partial s}=-2, \quad \frac{\partial y}{\partial s}=2, \quad \frac{\partial x}{\partial t}=3, \quad \frac{\partial y}{\partial t}=-1\)

12.6: Directional Derivatives

Terms and Concepts

1. What is the difference between a directional derivative and a partial derivative?

2. For what \(\vec{u}\) is \(D_{\vec{u}}f=f_x\)?

3. For what \(\vec{u}\) is \(D_{\vec{u}}f=f_y\)?

4. The gradient is _______ to level curves.

5. The gradient points in the direction of _______ increase.

6. It is generally more informative to view the directional derivative not as the result of a limit, but rather as the result of a _______ product.

Problems

In Exercises 7-12, a function \((z=f(x,y))\) is given. Find \(\nabla f\).

7. \((f(x,y)=-x^2y+xy^2+xy)\)

8. \((f(x,y)=\sin x \cos y)\)
9. \( f(x,y) = \frac{1}{x^2+y^2+1} \)

10. \( f(x,y) = 4x+3y \)

11. \( f(x,y) = x^2+2y^2-3xy-7x \)

12. \( f(x,y) = x^2y^3-2x \)

In Exercises 13-18, a function \( z=f(x,y) \) and a point \( P \) are given. Find the directional derivative of \( f \) in the indicated directions. Note: these are the same functions as in Exercises 7-12.

13. \( f(x,y) = -x^2y+xy^2+xy, \, P=(2,1) \)
   (a) In the direction of \( \langle \vec{v} \rangle = \langle 3,4 \rangle \)
   (b) In the direction toward the point \( Q=(-1,1) \)

14. \( f(x,y) = \sin x \cos y, \, P=(\frac{\pi}{4},\frac{\pi}{3}) \)
   (a) In the direction of \( \langle \vec{v} \rangle = \langle 1,1 \rangle \)
   (b) In the direction toward the point \( Q=(0,0) \)

15. \( f(x,y) = x^2+2y^2+1, \, P=(1,1) \)
   (a) In the direction of \( \langle \vec{v} \rangle = \langle 1,-1 \rangle \)
   (b) In the direction toward the point \( Q=(-2,-2) \)

16. \( f(x,y) = -4x+3y, \, P=(5,2) \)
   (a) In the direction of \( \langle \vec{v} \rangle = \langle 3,1 \rangle \)
   (b) In the direction toward the point \( Q=(2,7) \)

17. \( f(x,y) = x^2+2y^2-xy-7x, \, P=(4,1) \)
   (a) In the direction of \( \langle \vec{v} \rangle = \langle -2,5 \rangle \)
   (b) In the direction toward the point \( Q=(4,0) \)

18. \( f(x,y) = x^2y^3-2x, \, P=(1,1) \)
   (a) In the direction of \( \langle \vec{v} \rangle = \langle 3,3 \rangle \)
   (b) In the direction toward the point \( Q=(1,2) \)

In Exercises 19-24, a function \( z=f(x,y) \) and a point \( P \) are given.

(a) Find the direction of maximal increase of \( f \) at \( P \).
(b) What is the maximal value of \( |D_{\vec{u}}f| \) at \( P \)?
(c) Find the direction of minimal increase of \( f \) at \( P \).
(d) Give a direction \( \vec{u} \) such that \( |D_{\vec{u}}f=0| \) at \( P \).

Note: these are the same functions and points as in Exercises 13 through 18.

19. \( f(x,y) = -x^2y+xy^2+xy, \, P=(2,1) \)
20. \(f(x,y)=\sin x \cos y, \quad P=(\frac{\pi}{4}, \frac{\pi}{3})\)

21. \(f(x,y)=\frac{1}{x^2+y^2+1}, \quad P=(1,1)\)

22. \(f(x,y)=-4x+3y, \quad P=(5,4)\)

23. \(f(x,y)=x^2+y^2-xy-7x, \quad P=(4,1)\)

24. \(f(x,y)=x^2y^3-2x, \quad P=(1,1)\)

In Exercises 25-28, a function \(w=F(x,y,z)\), a vector \(\vec{v}\) and a point \(P\) are given.

(a) Find \(\nabla F(x,y,z)\).

(b) Find \(D_\vec{u}F\) at \(P\).

25. \(f(x,y)=3x^2z^3+4xy-3z^2, \quad \vec{v}=\langle 1,1,1 \rangle, \quad P=(3,2,1)\)

26. \(f(x,y)=\sin (x) \cos (y)e^z, \quad \vec{v}=\langle 2,2,1 \rangle, \quad P=(0,0,0)\)

27. \(f(x,y)=x^2y^2-y^2z^2, \quad \vec{v}=\langle -1,7,3 \rangle, \quad P=(1,0,-1)\)

28. \(f(x,y)=\frac{2}{x^2+y^2+z^2}, \quad \vec{v}=\langle 1,1,-2 \rangle, \quad P=(1,1,1)\)

12.7: Tangent Lines, Normal Lines, and Tangent Planes

Terms and Concepts

1. Explain how the vector \(\langle \vec{v} \rangle=\langle 1,0,3 \rangle\) can be thought of as having a "slope" of 3.

2. Explain how the vector \(\langle \vec{v} \rangle=\langle 0.6,0.8,-2 \rangle\) can be thought of as having a "slope" of -2.

3. T/F: Let \((z=f(x,y))\) be differentiable at \(P\). If \(\langle \vec{n} \rangle\) is normal vector to the tangent plane of \((f)\) at \((P)\), then \(\langle \vec{n} \rangle\) is orthogonal to \((f_x)\) and \((f_y)\) at \(P\).

4. Explain in your own words why we do not refer to the tangent line to a surface at a point, but rather to directional tangent lines to a surface at a point.

Problems

In Exercises 5-8, a function \((z=f(x,y))\), a vector \(\langle \vec{v} \rangle\) and a point \(P\) are given. Give the parametric equations of the following directional tangent lines to \((f)\) at \((P)\):

(a) \((l_x (t))\)

(b) \((l_y (t))\)
(c) \( l_\vec{u} \), where \( \vec{u} \) is the unit vector in the direction of \( \vec{v} \).

5. \( f(x,y) = 2x^y - 4xy^2, \quad \vec{v} = \langle 1, 1 \rangle, \quad P = (2, 3) \).

6. \( f(x,y) = 3\cos x \sin y, \quad \vec{v} = \langle 1, 2 \rangle, \quad P = (\pi/3, \pi/6) \).

7. \( f(x,y) = 3x - 5y, \quad \vec{v} = \langle 1, 1 \rangle, \quad P = (4, 2) \).

8. \( f(x,y) = x^2 - 2x - y^2 + 4y, \quad \vec{v} = \langle 1, 1 \rangle, \quad P = (1, 2) \).

In Exercises 9-12, a function \( f(x,y) \) and a point \( P \) are given. Find the equation of the normal line to \( f \) at \( P \).

Note: these are the same functions as in Exercises 5-8.

9. \( f(x,y) = 2x^y - 4xy^2, \quad \vec{v} = \langle 1, 3 \rangle, \quad P = (2, 3) \).

10. \( f(x,y) = 3\cos x \sin y, \quad \vec{v} = \langle 1, 2 \rangle, \quad P = (\pi/3, \pi/6) \).

11. \( f(x,y) = 3x - 5y, \quad \vec{v} = \langle 1, 1 \rangle, \quad P = (4, 2) \).

12. \( f(x,y) = x^2 - 2x - y^2 + 4y, \quad \vec{v} = \langle 1, 1 \rangle, \quad P = (1, 2) \).

In Exercises 13-16, a function \( f(x,y) \) and a point \( P \) are given. Find the two points that are 2 units from the surface \( f \) at \( P \).

Note: these are the same functions as in Exercises 5-8.

13. \( f(x,y) = 2x^y - 4xy^2, \quad \vec{v} = \langle 1, 3 \rangle, \quad P = (2, 3) \).

14. \( f(x,y) = 3\cos x \sin y, \quad \vec{v} = \langle 1, 2 \rangle, \quad P = (\pi/3, \pi/6) \).

15. \( f(x,y) = 3x - 5y, \quad \vec{v} = \langle 1, 1 \rangle, \quad P = (4, 2) \).

16. \( f(x,y) = x^2 - 2x - y^2 + 4y, \quad \vec{v} = \langle 1, 1 \rangle, \quad P = (1, 2) \).

In Exercises 17-20, a function \( f(x,y) \) and a point \( P \) are given. Find the equation of the tangent plane to \( f \) at \( P \).

Note: these are the same functions as in Exercises 5-8.

17. \( f(x,y) = 2x^y - 4xy^2, \quad \vec{v} = \langle 1, 3 \rangle, \quad P = (2, 3) \).

18. \( f(x,y) = 3\cos x \sin y, \quad \vec{v} = \langle 1, 2 \rangle, \quad P = (\pi/3, \pi/6) \).

19. \( f(x,y) = 3x - 5y, \quad \vec{v} = \langle 1, 1 \rangle, \quad P = (4, 2) \).

20. \( f(x,y) = x^2 - 2x - y^2 + 4y, \quad \vec{v} = \langle 1, 1 \rangle, \quad P = (1, 2) \).

In Exercises 21-24, an implicitly defined function of \( (x,y) \) and \( (z) \) is given along with a point \( P \) that lies on the surface. Use the gradient \( \nabla F \) to:

(a) find the equation of the normal line to the surface at \( P \), and
(b) find the equation of the plane tangent to the surface at $P$.

21. \( \frac{x^2}{8} + \frac{y^2}{4} + \frac{z^2}{16} = 1 \text{ at } P = (1, \sqrt{2}, \sqrt{6}) \)

22. \( z^2 - \frac{x^2}{4} - \frac{y^2}{9} = 0 \text{ at } P = (4, -3, \sqrt{5}) \)

23. \( xy^2 - xz^2 = 0 \text{ at } P = (2, 1, -1) \)

24. \( (\sin(xy) + \cos(yz)) = 0 \text{ at } P = (2, \pi/12, 4) \)

### 12.8: Extreme Values

**Terms and Concepts**

1. T/F: Theorem 114 states that if \( f \) has a critical point at \( P \), then \( f \) has a relative extrema at \( P \).

2. T/F: A point \( P \) is a critical point of \( f \) if \( f_x \) and \( f_y \) are both 0 at \( P \).

3. T/F: A point \( P \) is a critical point of \( f \) if \( f_x \) or \( f_y \) are undefined at \( P \).

4. Explain what it means to "solve a constrained optimization" problem.

**Problems**

**Exercises 5-14**, find the critical points of the given function. Use the Second Derivative Test to determine if each critical point corresponds to a relative maximum, minimum, or saddle point.

5. \( f(x,y) = \frac{1}{2}x^2 + 2y^2 - 8y + 4x \)

6. \( f(x,y) = x^2 + 4x + y^2 - 9y + 3xy \)

7. \( f(x,y) = x^2 + 3y^2 - 6y + 4xy \)

8. \( f(x,y) = \frac{1}{2}x^2 + y^2 + 1 \)

9. \( f(x,y) = x^2 + y^2 - 3y + 1 \)

10. \( f(x,y) = 3x - x^3 + 3y^3 - 3y \)

11. \( f(x,y) = x^2y^2 \)

12. \( f(x,y) = x^4 - 2x^2 + y^3 - 27y - 15 \)

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13. \( f(x,y) = \sqrt{16-(x-3)^2-y^2} \)

14. \( f(x,y) = \sqrt{x^2+y^2} \)

In Exercises 15-18, find the absolute maximum and minimum of the function subject to the given constraint.

15. \( f(x,y) = x^2+y^2+y=1 \), constrained to the triangles with vertices \( (0,1), (-1,1), \text{ and } (1,-1) \).

16. \( f(x,y) = 5x-7y \), constrained to the region bounded by \( y=x^2 \text{ and } y=1 \).

17. \( f(x,y) = x^2+2x+y^2+2y \), constrained to the region bounded by the circle \( x^2+y^2=4 \).

18. \( f(x,y) = 3y-2x^2 \), constrained to the region bounded by the parabola \( y=x^2+x-1 \) and the line \( y=x \).