Module 1: Problem Solving Strategies

Unlike exercises, there is never a simple recipe for solving a problem. You can get better and better at solving problems, both by building up your background knowledge and by simply practicing. As you solve more problems (and learn how other people solved them), you learn strategies and techniques that can be useful. But no single strategy works every time.

Pólya’s *How to Solve It*

George Pólya was a great champion in the field of teaching effective problem solving skills. He was born in Hungary in 1887, received his Ph.D. at the University of Budapest, and was a professor at Stanford University (among other universities). He wrote many mathematical papers along with three books, most famously, “How to Solve it.” Pólya died at the age 98 in 1985.1

1. Image of Pólya by Thane Plambeck from Palo Alto, California (Flickr) [CC BY
In 1945, Pólya published the short book *How to Solve It*, which gave a four-step method for solving mathematical problems:

1. First, you have to understand the problem.
2. After understanding, then make a plan.
3. Carry out the plan.
4. Look back on your work. How could it be better?

This is all well and good, but how do you actually do these steps?!?! Steps 1. and 2. are particularly mysterious! How do you “make a plan?” That is where you need some tools in your toolbox, and some experience to draw upon.

Much has been written since 1945 to explain these steps in more detail, but the truth is that they are more art than science. This is where math becomes a creative endeavor (and where it becomes so much fun). We will articulate some useful problem solving strategies, but no such list will ever be complete. This is really just a start to help you on your way. The best way to become a skilled problem solver is to learn the background material well, and then to solve a lot of problems!

**Problem Solving Strategy 1 (Guess and Test)**

Make a guess and test to see if it satisfies the demands of the problem. If it doesn't, alter the guess appropriately and check again. Keep doing this until you find a solution.

**Example:**

Mr. Jones has a total of 25 chickens and cows on his farm. How many of each does he have if all together there are 76 feet?

**Step 1: Understanding the problem**
We are given in the problem that there are 25 chickens and cows. All together there are 76 feet. Chickens have 2 feet and cows have 4 feet. We are trying to determine how many cows and how many chickens Mr. Jones has on his farm.

**Step 2: Devise a plan**

Going to use Guess and test along with making a tab

Many times the strategy below is used with guess and test.

Make a table and look for a pattern:

Procedure: Make a table reflecting the data in the problem. If done in an orderly way, such a table will often reveal patterns and relationships that suggest how the problem can be solved.

**Step 3: Carry out the plan:**

<table>
<thead>
<tr>
<th>Chickens</th>
<th>Cows</th>
<th>Number of chicken feet</th>
<th>Number of cow feet</th>
<th>Total number of feet</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>5</td>
<td>40</td>
<td>20</td>
<td>60</td>
</tr>
<tr>
<td>21</td>
<td>4</td>
<td>42</td>
<td>16</td>
<td>58</td>
</tr>
</tbody>
</table>

Notice we are going in the wrong direction! The total number of feet is decreasing!

| 19       | 6    | 38                     | 24                | 62                  |

Better! The total number of feet are increasing!

<table>
<thead>
<tr>
<th>15</th>
<th>10</th>
<th>30</th>
<th>40</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>13</td>
<td>24</td>
<td>52</td>
<td>76</td>
</tr>
</tbody>
</table>

**Step 4: Looking back:**

Check: 12 + 13 = 25 heads

24 + 52 = 76 feet.

We have found the solution to this problem. I could use this strategy when there are a limited number of possible answers and when two items are the same but they have one characteristic that is different.

**Videos to watch:**

1. Click on this link to see an example of “Guess and Test”

2. Click on this link to see another example of Guess and Test.

Check in question 1:

Place the digits 8, 10, 11, 12, and 13 in the circles to make the sums across and vertically equal 31. (5 points)

Check in question 2:

Old McDonald has 250 chickens and goats in the barnyard. Altogether there are 760 feet. How many of each animal does he have? Make sure you use Polya’s 4 problem solving steps. (12 points)

Problem Solving Strategy 2 (Draw a Picture). Some problems are obviously about a geometric situation, and it is clear you want to draw a picture and mark down all of the given information before you try to solve it. But even for a problem that is not geometric thinking visually can help!

Videos to watch demonstrating how to use "Draw a Picture".
1. Click on this link to see an example of “Draw a Picture”
http://www.mathstories.com/strategies.htm
2. Click on this link to see another example of Draw a Picture.

Problem Solving Strategy 3 (Using a variable to find the sum of a sequence.)

Gauss's strategy for sequences.

last term = fixed number \((n-1)\) + first term

The fix number is the the amount each term is increasing or decreasing by. "n" is the number of terms you have. You can use this formula to find the last term in the sequence or the number of terms you have in a sequence.

Ex: 2, 5, 8, ... Find the 200th term.
Last term = 3(200-1) +2
Last term is 599.

To find the sum of a sequence: sum = \([(first\ term + last\ term) \ (number\ of\ terms)]/\ 2
Sum = (2 + 599) (200) then divide by 2
Sum = 60,100

**Check in question 3:** (10 points)
Find the 320\textsuperscript{th} term of 7, 10, 13, 16 …
Then find the sum of the first 320 terms.

**Problem Solving Strategy 4 (Working Backwards)**

This is considered a strategy in many schools. If you are given an answer, and the steps that were taken to arrive at that answer, you should be able to determine the starting point.

**Videos to watch demonstrating of “Working Backwards”**

http://www.mathstories.com/strategies.htm
https://www.youtube.com/watch?v=5FFWTsMFeJw

**Example:**
Karen is thinking of a number. If you double it, and subtract 7, you obtain 11. What is Karen’s number?
1. We start with 11 and work backwards.
2. The opposite of subtraction is addition. We will add 7 to 11. We are now at 18.
3. The opposite of doubling something is dividing by 2. 18/2 = 9
4. This should be our answer. Looking back:
   
   9 x 2 = 18 -7 = 11
5. We have the right answer.

**Check in question 4:**
Christina is thinking of a number.
If you multiply her number by 93, add 6, and divide by 3, you obtain 436. What is her number? Solve this problem by working backwards. (5 points)

**Problem Solving Strategy 5 (Looking for a Pattern)**

Definition: A sequence is a pattern involving an ordered arrangement of numbers.

We first need to find a pattern.

Ask yourself as you search for a pattern – are the numbers growing steadily larger? Steadily smaller? How is each number related?

**Example 1:** 1, 4, 7, 10, 13…
Find the next 2 numbers. The pattern is each number is increasing by 3. The next two numbers would be 16 and 19.

**Example 2:** 1, 4, 9, 16 … find the next 2 numbers. It looks like each successive number is increase by the next odd number. $1 + 3 = 4$.

$4 + 5 = 9$

$9 + 7 = 16$

So the next number would be

$16 + 9 = 25$

$25 + 11 = 36$

**Example 3:** 10, 7, 4, 1, -2… find the next 2 numbers.

In this sequence, the numbers are decreasing by 3. So the next 2 numbers would be $-2 - 3 = -5$

$-5 – 3 = -8$

**Example 4:** 1, 2, 4, 8 … find the next two numbers.

This example is a little bit harder. The numbers are increasing but not by a constant. Maybe a factor?

$1 \times 2 = 2$

$2 \times 2 = 4$

$4 \times 2 = 8$

So each number is being multiplied by 2.

$8 \times 2 = 16$

$16 \times 2 = 32$

**Videos:**

1. Click on this link to see an example of “Looking for a Pattern”


2. Click on this link to see another example of Looking for a Pattern.


**Problem Solving Strategy 6 (Make a List)**

**Example 1:** Can perfect squares end in a 2 or a 3?

List all the squares of the numbers 1 to 20.

$1 4 9 16 25 36 49 64 81 100 121 144 169 196 225 256 289 324 361 400$.

Now look at the number in the ones digits. Notice they are 0, 1, 4, 5, 6, or 9. Notice none of the perfect squares
end in 2, 3, 7, or 8. This list suggests that perfect squares cannot end in a 2, 3, 7 or 8.

**Example 2:**

How many different amounts of money can you have in your pocket if you have only three coins including only dimes and quarters?

Quarter’s dimes

<table>
<thead>
<tr>
<th>0</th>
<th>3</th>
<th>30 cents</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>45 cents</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>60 cents</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>75 cents</td>
</tr>
</tbody>
</table>

**Videos demonstrating "Make a List"**

http://www.mathstories.com/strategies.htm


**Check in question 5:**

How many ways can you make change for 23 cents using only pennies, nickels, and dimes? (10 points)

**Problem Solving Strategy 7 (Solve a Simpler Problem)**

**Example:**

Geometric Sequences:

How would we find the nth term?

Solve a simpler problem:

1, 3, 9, 27.

1. To get from 1 to 3 what did we do?
2. To get from 3 to 9 what did we do?

Let’s set up a table:

<table>
<thead>
<tr>
<th>Term Number</th>
<th>what did we do</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 * 3</td>
</tr>
<tr>
<td>2</td>
<td>3 * 3</td>
</tr>
<tr>
<td>3</td>
<td>9 * 3</td>
</tr>
</tbody>
</table>

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Looking back: How would you find the nth term?

Nth term = 1 times

Find the 10th term of the above sequence.

Let L = the tenth term

L = 1 times

= 19,683

**Problem Solving Strategy 8 (Process of Elimination)**

This strategy can be used when there is only one possible solution.

**Example:**

I’m thinking of a number.

The number is odd.

It is more than 1 but less than 100.

It is greater than 20.

It is less than 5 times 7.

The sum of the digits is 7.

It is evenly divisible by 5.

a. We know it is an odd number between 1 and 100.

b. It is greater than 20 but less than 35

21, 23, 25, 27, 29, 31, 33, 35. These are the possibilities.

c. The sum of the digits is 7

21 (2+1=3) No 23 (2+3 = 5) No 25 (2 + 5= 7) Yes Using the same process we see there are no other numbers that meet this criteria. Also we notice 25 is divisible by 5. By using the strategy elimination, we have found our answer.

**Check in question 6:** (8 points)

Jose is thinking of a number.

The number is not odd.
The sum of the digits is divisible by 2.
The number is a multiple of 11.
It is greater than 5 times 4.
It is a multiple of 6
It is less than 7 times 8 + 23
What is the number?

Click on this link for a quick review of the problem solving strategies.