1.8: Limits and continuity of Inverse Trigonometric functions

Inverse functions

Recall that a function \( f \) is one-to-one (often written as \( 1-1 \)) if it assigns distinct values of \( y \) to distinct values of \( x \). In other words, if \( (x_1 \ne x_2) \) then \( f(x_1) \ne f(x_2) \). Equivalently, \( f \) is one-to-one if \( f(x_1) = f(x_2) \) implies \( x_1 = x_2 \). There is a simple horizontal rule for determining whether a function \( y=f(x) \) is one-to-one: \( f \) is one-to-one if and only if every horizontal line intersects the graph of \( y=f(x) \) in the \( xy \)-coordinate plane at most once (see Figure 5.3.3).

![Horizontal rule for one-to-one functions](image)

**Figure 5.3.3** Horizontal rule for one-to-one functions

If a function \( f \) is one-to-one on its domain, then \( f \) has an inverse function, denoted by \( f^{-1} \), such that \( y=f(x) \) if and only if \( f^{-1}(y) = x \). The domain of \( f^{-1} \) is the range of \( f \).

The basic idea is that \( f^{-1} \) "undoes" what \( f \) does, and vice versa. In other words,
\( f(f^{-1}(y)) \cong y \quad \text{for all } y \text{ in the range of } f. \) 

Theorem \( \PageIndex{1} \)

If \( f \) is continuous and one to one, then \( f^{-1} \) is continuous on its domain.

**Inverse Trigonometric functions**

We know from their graphs that none of the trigonometric functions are one-to-one over their entire domains. However, we can restrict those functions to *subsets* of their domains where they *are* one-to-one. For example, \( y = \sin x \) is one-to-one over the interval \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \), as we see in the graph below:

![Graph of sin(x)](image)

For \( \left( -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \right) \) we have \( \left( -1 \leq \sin x \leq 1 \right) \), so we can define the **inverse sine** function \( y = \sin^{-1}(x) \) (sometimes called the **arc sine** and denoted by \( y = \arcsin(x) \)) whose domain is the interval \( \left[ -1, 1 \right] \) and whose range is the interval \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \). In other words:

\[
\begin{alignat}{3}
\sin^{-1} (\sin y) &\cong y \quad \text{for } -\tfrac{\pi}{2} \leq y \leq \tfrac{\pi}{2} \\
\sin (\sin^{-1} x) &\cong x \quad \text{for } -1 \leq x \leq 1
\end{alignat}
\]

**Summary of Inverse Trigonometric functions**

Let's illustrate the summary of Trigonometric functions and Inverse Trigonometric functions in following table:

<table>
<thead>
<tr>
<th>Trigonometric function</th>
<th>graph of the Trigonometric function</th>
<th>Restricted domain and</th>
<th>Inverse Trigonometric function</th>
</tr>
</thead>
</table>

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The range

\( f(x) = \sin(x) \)

\( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \)

\( \left[ -1, 1 \right] \)

\( f^{-1}(x) = \sin^{-1} x \)

\( \left( -1, 1 \right) \)

\( f^{-1}(-1) = \sin^{-1}(-1) \)

\( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \)

\( f^{-1}(-1) = \sin^{-1}(-1) \)

\( \left[ -1, 1 \right] \)

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\( f^{-1}(-1) = \sin^{-1}(-1) \)

\( \left[ -1, 1 \right] \)
\[
f(x) = \tan(x)
\]
and \[
f^{-1}(x) = \tan^{-1} x
\] for \(-\frac{\pi}{2}, \frac{\pi}{2}\) and \(\mathbb{R}\).
\( f(x) = \sec(x) \) on \([0, \pi]\), with \( x \neq \frac{\pi}{2} \) and \( \mathbb{R} \) below.

Below are examples:

Example \( \PageIndex{1} \):

Find \( \sin^{-1} \left( \sin \left( \frac{\pi}{4} \right) \right) \).

Solution:

Since \( -\frac{\pi}{2} \leq \frac{\pi}{4} \leq \frac{\pi}{2} \), we know that \( \sin^{-1} \left( \sin \left( \frac{\pi}{4} \right) \right) = \boxed{\frac{\pi}{4}} \), by Equation \( \ref{eqn:arcsin1} \).

Example \( \PageIndex{2} \):

Find \( \sin^{-1} \left( \sin \left( \frac{5\pi}{4} \right) \right) \).

Solution:

Since \( -\frac{\pi}{2} \leq \frac{5\pi}{4} \leq \frac{3\pi}{2} \), we cannot use Equation \( \ref{eqn:arcsin1} \). But we know that \( \sin \left( \frac{5\pi}{4} \right) = -\frac{\sqrt{2}}{2} \). Thus, \( \sin^{-1} \left( \sin \left( \frac{5\pi}{4} \right) \right) = \sin^{-1} \left( -\frac{\sqrt{2}}{2} \right) \) is, by definition, the angle \( y \) such that \( -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \) and \( \sin y = -\frac{\sqrt{2}}{2} \). That angle is...
Example (PageIndex{3}):
Find \( \cos^{-1} \left(\cos\frac{\pi}{3}\right) \).

**Solution:**

Since \(0 \leq \frac{\pi}{3} \leq \pi\), we know that \( \cos^{-1} \left(\cos\frac{\pi}{3}\right) = \boxed{\frac{\pi}{3}} \), by Equation \ref{eqn:arccos1}.

Example (PageIndex{4}):
Find \( \cos^{-1} \left(\cos\frac{4\pi}{3}\right) \).

**Solution:**

Since \(\frac{4\pi}{3} > \pi\), we can not use Equation \ref{eqn:arccos1}. But we know that \( \cos\frac{4\pi}{3} = -\frac{1}{2} \). Thus, \( \cos^{-1} \left(\cos\frac{4\pi}{3}\right) = \boxed{\frac{2\pi}{3}} \) (i.e. \(120^\circ\)).

Example (PageIndex{5}):
Find \( \tan^{-1} \left(\tan\frac{\pi}{4}\right) \).

**Solution:**

Since \(\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\), we know that \( \tan^{-1} \left(\tan\frac{\pi}{4}\right) = \boxed{\frac{\pi}{4}} \), by Equation \ref{eqn:arctan1}.

Example (PageIndex{6}):
Find \( \tan^{-1} \left(\tan\pi\right) \).

**Solution:**

Since \(\pi \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\), we can not use Equation \ref{eqn:arctan1}. But we know that \( \tan\pi = 0 \). Thus, \( \tan^{-1} \left(\tan\pi\right) = \boxed{0} \).
Find the exact value of \( \cos \left( \sin^{-1} \left( -\frac{1}{4} \right) \right) \).

**Solution:**

Let \( \theta = \sin^{-1} \left( -\frac{1}{4} \right) \). We know that \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \), so since \( \sin \theta = -\frac{1}{4} < 0 \), \( \theta \) must be in QIV. Hence \( \cos \theta > 0 \). Thus,

\[
\cos^2 \theta = 1 - \sin^2 \theta = 1 - \left( -\frac{1}{4} \right)^2 = \frac{15}{16}
\]

\[
\Rightarrow \cos \theta = \frac{\sqrt{15}}{4}.
\]

Note that we took the positive square root above since \( \cos \theta > 0 \). Thus, \( \cos \left( \sin^{-1} \left( -\frac{1}{4} \right) \right) = \boxed{\frac{\sqrt{15}}{4}} \).

Example \( \PageIndex{8} \):

Show that \( \tan \left( \sin^{-1} x \right) = \frac{x}{\sqrt{1 - x^2}} \) for \( -1 < x < 1 \).

**Solution:**

When \( x = 0 \), the Equation holds trivially, since

\[
\tan \left( \sin^{-1} 0 \right) = \tan 0 = 0 = \frac{0}{\sqrt{1 - 0^2}}.
\]

Now suppose that \( 0 < x < 1 \). Let \( \theta = \sin^{-1} x \). Then \( \theta \) is in QI and \( \sin \theta = x \). Draw a right triangle with an angle \( \theta \) such that the opposite leg has length \( x \) and the hypotenuse has length \( 1 \), as in Figure 5.3.10 (note that this is possible since \( 0 < x < 1 \)). Then \( \sin \theta = \frac{x}{1} = x \). By the Pythagorean Theorem, the adjacent leg has length \( \sqrt{1 - x^2} \). Thus, \( \tan \theta = \frac{x}{\sqrt{1 - x^2}} \).

If \( -1 < x < 0 \) then \( \theta = \sin^{-1} x \) is in QIV. So we can draw the same triangle except that it would be "upside down" and we would again have \( \tan \theta = \frac{x}{\sqrt{1 - x^2}} \), since the tangent and sine have the same sign (negative) in QIV. Thus, \( \tan \left( \sin^{-1} x \right) = \frac{x}{\sqrt{1 - x^2}} \) for \( -1 < x < 1 \).
Example \(\PageIndex{9}\):

Prove the identity \(\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\).

Solution:

Let \(\theta = \cot^{-1} x\). Using relations, we have

\[
\tan\left( \frac{\pi}{2} - \theta \right) = -\tan\left( \theta - \frac{\pi}{2} \right) = \cot \theta = \cot(\cot^{-1} x) = x, \tag{Eqn:arccot2}
\]

by Equation \(\ref{eqn:arccot2}\). So since \(\tan(\tan^{-1} x) = x\) for all \(x\), this means that \(\tan\left( \frac{\pi}{2} - \theta \right) = \tan\left( \theta - \frac{\pi}{2} \right)\). Thus, \(\tan\left( \frac{\pi}{2} - \theta \right) = \tan\left( \theta - \frac{\pi}{2} \right)\).

Now, we know that \(0 < \cot^{-1} x < \pi\), so \(-\frac{\pi}{2} < \frac{\pi}{2} - \cot^{-1} x < \frac{\pi}{2}\), i.e. \(\frac{\pi}{2} - \cot^{-1} x\) is in the restricted subset on which the tangent function is one-to-one. Hence, \(\tan\left( \frac{\pi}{2} - \cot^{-1} x \right)\) implies that \(\tan^{-1} x = \frac{\pi}{2} - \cot^{-1} x\), which proves the identity.

### Continuity of Inverse Trigonometric functions

Example \(\PageIndex{1}\):

Let \(f(x) = \frac{3 \sec^{-1} (x)}{4 - \tan^{-1} (x)}\). Find the values (if any) for which \(f(x)\) is continuous.

Exercise \(\PageIndex{1}\)

Let \(f(x) = \frac{3 \sec^{-1} (x)}{8 + 2 \tan^{-1} (x)}\). Find the values (if any) for which \(f(x)\) is continuous.

**Answer**
Limit of Inverse Trigonometric functions

Theorem \(\PageIndex{1}\)

\[
\lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2}.
\]

\[
\lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}.
\]

\[
\lim_{x \to \infty} \sec^{-1}(x) = \lim_{x \to \infty} \sec^{-1}(x) = \frac{\pi}{2}.
\]

Example \(\PageIndex{1}\):

Find \(\lim_{x \to \infty} \sin(2 \tan^{-1}(x))\).

Exercise \(\PageIndex{1}\)

Find \(\lim_{x \to -\infty} \sin(2 \tan^{-1}(x))\).

Answer

Contributors

- Michael Corral (Schoolcraft College). The content of this page is distributed under the terms of the GNU Free Documentation License, Version 1.2.

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- Pamini Thangarajah (Mount Royal University, Calgary, Alberta, Canada)