2.8: Limits and continuity of Inverse Trigonometric functions

Inverse functions

Recall that a function f is **one-to-one** (often written as $1-1$) if it assigns distinct values of y to distinct values of x. In other words, if $x_1 \ne x_2$ then $f(x_1) \ne f(x_2)$. Equivalently, f is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. There is a simple *horizontal rule* for determining whether a function $y=f(x)$ is one-to-one: f is one-to-one if and only if every horizontal line intersects the graph of $y=f(x)$ in the xy-coordinate plane at most once (see Figure 5.3.3).

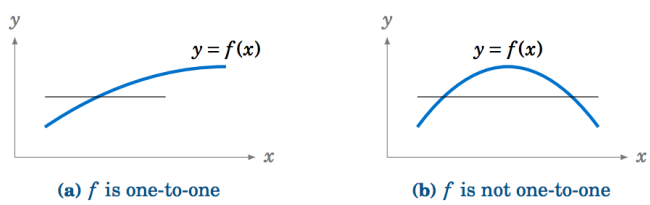


Figure 5.3.3 Horizontal rule for one-to-one functions

If a function f is one-to-one on its domain, then f has an **inverse function**, denoted by f^{-1}, such that $y=f(x)$ if and only if $f^{-1}(y) = x$. The domain of f^{-1} is the range of f.

The basic idea is that f^{-1} "undoes" what f does, and vice versa. In other words,
$$\nonumber \begin{alignat*}{3}
f^{-1}(f(x)) ~&=~ x \quad&&\text{for all x in the domain of f, and}\\ \nonumber
$$

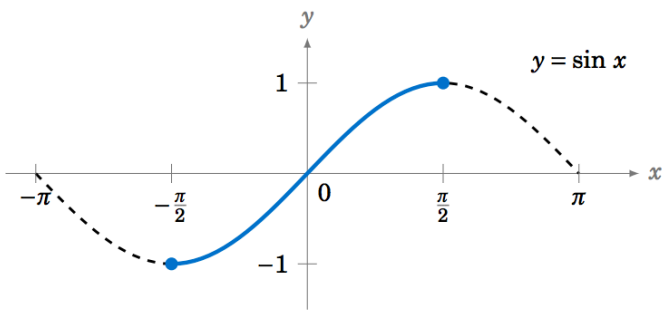
f(f^{-1}(y)) ~&=~ y \quad&&\text{for all \(y \) in the range of \(f \).}
\end{alignat*}\]

Theorem \(\PageIndex{1}\)

If \(f\) is continuous and one to one, then \(f^{-1}\ is continuous on its domain.

Inverse Trigonometric functions

We know from their graphs that none of the trigonometric functions are one-to-one over their entire domains. However, we can restrict those functions to *subsets* of their domains where they *are* one-to-one. For example, \(y=\sin\;x \) is one-to-one over the interval \(\left[-\frac{\pi}{2},\frac{\pi}{2} \right] \), as we see in the graph below:



For \(-\frac{\pi}{2} \le x \le \frac{\pi}{2} \) we have \(-1 \le \sin\;x \le 1 \), so we can define the **inverse sine** function \(y=\sin^{-1} x \) (sometimes called the **arc sine** and denoted by \(y=\arcsin\;(x\)) whose domain is the interval \([-1,1] \) and whose range is the interval \(\left[-\frac{\pi}{2},\frac{\pi}{2} \right] \). In other words:

\[\begin{alignat}{3}
\sin^{-1} (\sin\;y) ~&=~ y \quad&&\text{for \(-\tfrac{\pi}{2} \le y \le
\tfrac{\pi}{2}\)}\label{eqn:arcsin1}\\
\sin\;(\sin^{-1} x) ~&=~ x \quad&&\text{for \(-1 \le x \le 1\)}\label{eqn:arcsin2}
\end{alignat}\]

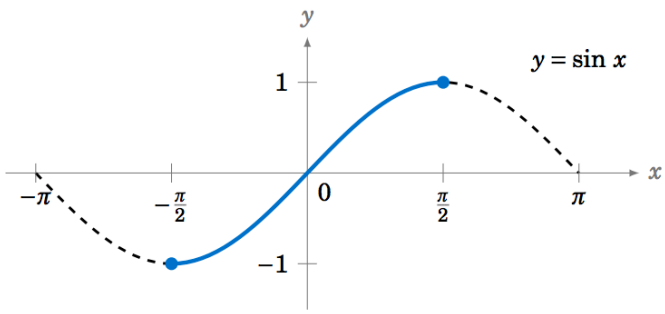
Summary of Inverse Trigonometric functions

Lets illustrate the summary of Trigonometric functions and Inverse Trigonometric functions in following table:

Trigonometric function	graph of the Trigonometric function	Restricted domain and	Inverse Trigonometric function

the range

\(f(x)=\sin(x)\)

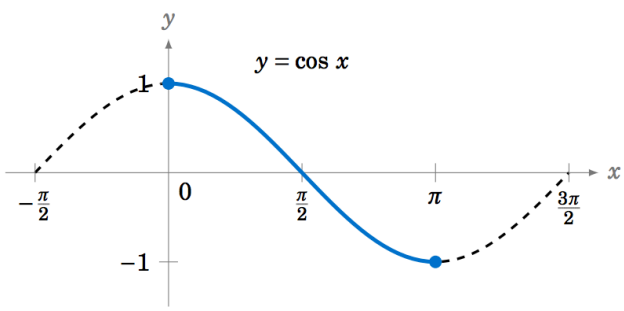


\(\left[-\frac{\pi}{2},\frac{\pi}{2} \right] \)

and \([-1,1]\)

\(f^{-1}(x)=\sin^{-1} x \)

\(f(x)=\cos(x)\)

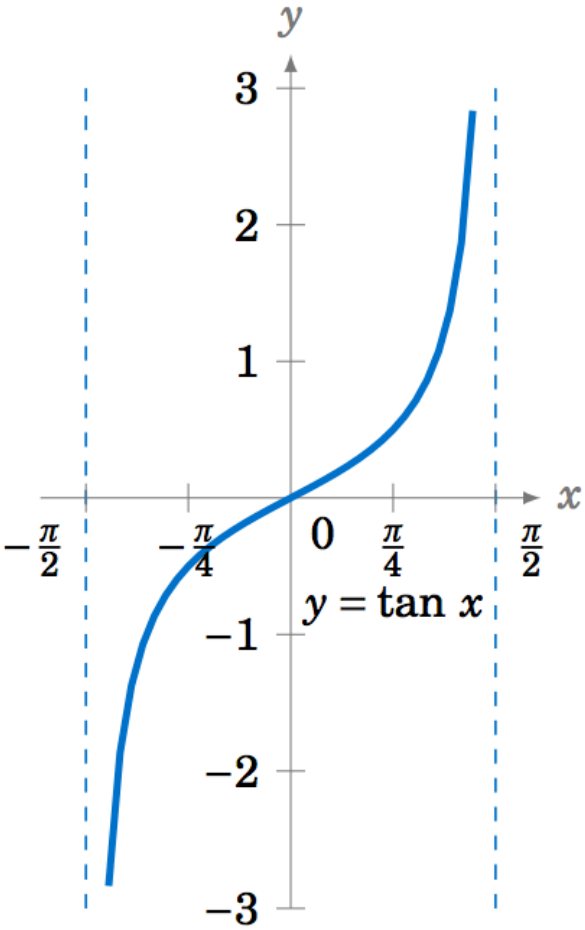


\([0,\pi]\)

and \([-1,1]\)

\(f^{-1}(x)=\cos^{-1} x \)

\(f(x)=\tan(x)\)



\(\left(
-\frac{\pi}{2},\frac{\pi}{2}
\right) \)

and \(\mathbb{R}\)

\(f^{-1}(x)=\tan^{-1}
x \)

\(f(x)=\cot(x)\)

$f(x)=\sec(x)$	$[0,\pi]$, with $x\ne \frac{\pi}{2}$ and $\mathbb{R}$

Below are examples:

Example $\PageIndex{1}$:

Find $\sin^{-1} \left(\sin\;\frac{\pi}{4}\right) $.

Solution

Since $-\frac{\pi}{2} \le \frac{\pi}{4} \le \frac{\pi}{2} $, we know that $\sin^{-1} \left(\sin\;\frac{\pi}{4}\right) = \boxed{\frac{\pi}{4}}\; $, by Equation \ref{eqn:arcsin1}.

Example $\PageIndex{2}$:

Find $\sin^{-1} \left(\sin\;\frac{5\pi}{4}\right) $.

Solution

Since $\frac{5\pi}{4} > \frac{\pi}{2} $, we can not use Equation \ref{eqn:arcsin1}. But we know that $\sin\;\frac{5\pi}{4} = -\frac{1}{\sqrt{2}} $. Thus, $\sin^{-1} \left(\sin\;\frac{5\pi}{4}\right) = \sin^{-1} \left(-\frac{1}{\sqrt{2}} \right) $ is, by definition, the angle $y $ such that $-\frac{\pi}{2} \le y \le \frac{\pi}{2} $ and $\sin\;y = -\frac{1}{\sqrt{2}} $. That angle is

$y=-\frac{\pi}{4}$, since

$$\sin\;\left(-\tfrac{\pi}{4} \right) ~=~ -\sin\;\left(\tfrac{\pi}{4} \right) ~=~ -\tfrac{1}{\sqrt{2}} ~. \nonumber$$

Example $\PageIndex{3}$:

Find $\cos^{-1} \left(\cos\;\frac{\pi}{3}\right)$.

Solution

Since $0 \le \frac{\pi}{3} \le \pi$, we know that $\cos^{-1} \left(\cos\;\frac{\pi}{3}\right) = \boxed{\frac{\pi}{3}}\;$, by Equation \ref{eqn:arccos1}.

Example $\PageIndex{4}$:

Find $\cos^{-1} \left(\cos\;\frac{4\pi}{3}\right)$.

Solution

Since $\frac{4\pi}{3} > \pi$, we can not use Equation \ref{eqn:arccos1}. But we know that $\cos\;\frac{4\pi}{3} = -\frac{1}{2}$. Thus, $\cos^{-1} \left(\cos\;\frac{4\pi}{3}\right) = \cos^{-1} \left(-\frac{1}{2} \right)$ is, by definition, the angle y such that $0 \le y \le \pi$ and $\cos\;y = -\frac{1}{2}$. That angle is $y=\frac{2\pi}{3}$ (i.e. 120°). Thus, $\cos^{-1} \left(\cos\;\frac{4\pi}{3}\right) = \boxed{\tfrac{2\pi}{3}}\;$.

Example $\PageIndex{5}$:

Find $\tan^{-1} \left(\tan\;\frac{\pi}{4}\right)$.

Solution

Since $-\tfrac{\pi}{2} \le \tfrac{\pi}{4} \le \tfrac{\pi}{2}$, we know that $\tan^{-1} \left(\tan\;\frac{\pi}{4}\right) = \boxed{\frac{\pi}{4}}\;$, by Equation \ref{eqn:arctan1}.

Example $\PageIndex{6}$:

Find $\tan^{-1} \left(\tan\;\pi\right)$.

Solution

Since $\pi > \tfrac{\pi}{2}$, we can not use Equation \ref{eqn:arctan1}. But we know that $\tan\;\pi = 0$. Thus, $\tan^{-1} \left(\tan\;\pi\right) = \tan^{-1} 0$ is, by definition, the angle y such that $-\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2}$ and $\tan\;y = 0$. That angle is $y=0$. Thus, $\tan^{-1} \left(\tan\;\pi \right) = \boxed{0}\;$.

Example$\PageIndex{7}$:

Find the exact value of $\cos\;\left(\sin^{-1}\;\left(-\frac{1}{4}\right)\right)$.

Solution

Let $\theta = \sin^{-1}\;\left(-\frac{1}{4}\right)$. We know that $-\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}$, so since $\sin\;\theta = -\frac{1}{4} < 0$, θ must be in QIV. Hence $\cos\;\theta > 0$. Thus,

$$\cos^2 \;\theta ~=~ 1 ~-~ \sin^2 \;\theta ~=~ 1 ~-~ \left(-\frac{1}{4} \right)^2 ~=~\frac{15}{16} \quad\Rightarrow\quad \cos\;\theta ~=~ \frac{\sqrt{15}}{4} ~. \nonumber$$

Note that we took the positive square root above since $\cos\;\theta > 0$. Thus, $\cos\;\left(\sin^{-1}\;\left(-\frac{1}{4}\right)\right) = \boxed{\frac{\sqrt{15}}{4}}\;$.

Example $\PageIndex{8}$:

Show that $\tan\;(\sin^{-1} x) = \dfrac{x}{\sqrt{1 - x^2}}$ for $-1 < x < 1$.

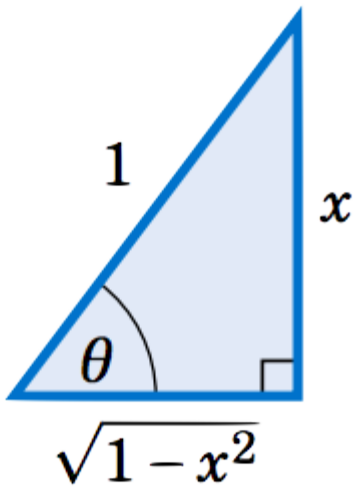
Solution

When $x=0$, the Equation holds trivially, since

$$\nonumber \tan\;(\sin^{-1} 0) ~=~ \tan\;0 ~=~ 0 ~=~ \dfrac{0}{\sqrt{1 - 0^2}} ~.$$

Now suppose that $0 < x < 1$. Let $\theta = \sin^{-1} x$. Then θ is in QI and $\sin\;\theta = x$. Draw a right triangle with an angle θ such that the opposite leg has length x and the hypotenuse has length 1, as in Figure 5.3.10 (note that this is possible since $0 < x < 1$). Then $\sin\;\theta = \frac{x}{1} = x$. By the Pythagorean Theorem, the adjacent leg has length $\sqrt{1 - x^2}$. Thus, $\tan\;\theta = \frac{x}{\sqrt{1 - x^2}}$.

If $-1 < x < 0$ then $\theta = \sin^{-1} x$ is in QIV. So we can draw the same triangle except that it would be "upside down" and we would again have $\tan\;\theta = \frac{x}{\sqrt{1 - x^2}}$, since the tangent and sine have the same sign (negative) in QIV. Thus, $\tan\;(\sin^{-1} x) = \dfrac{x}{\sqrt{1 - x^2}}$ for $-1 < x < 1$.



Example \(\PageIndex{9}\):

Prove the identity \(\tan^{-1} x \;+\; \cot^{-1} x ~=~ \frac{\pi}{2} \).

Solution:

Let \(\theta = \cot^{-1} x \). Using relations, we have

\[\nonumber \tan\;\left(\tfrac{\pi}{2} - \theta \right) ~=~ -\tan\;\left(\theta - \tfrac{\pi}{2} \right)
~=~ \cot\;\theta ~=~ \cot\;(\cot^{-1} x) ~=~ x ~,\]

by Equation \ref{eqn:arccot2}. So since \(\tan\;(\tan^{-1} x) = x \) for all \(x \), this means that \(\tan\;(\tan^{-1} x) = \tan\;\left(\tfrac{\pi}{2} - \theta \right) \). Thus, \(\tan\;(\tan^{-1} x) = \tan\;\left(\tfrac{\pi}{2} - \cot^{-1} x \right) \). Now, we know that \(0 < \cot^{-1} x < \pi \), so \(-\tfrac{\pi}{2} < \tfrac{\pi}{2} - \cot^{-1} x < \tfrac{\pi}{2} \), i.e. \(\tfrac{\pi}{2} - \cot^{-1} x \) is in the restricted subset on which the tangent function is one-to-one. Hence, \(\tan\;(\tan^{-1} x) = \tan\;\left(\tfrac{\pi}{2} - \cot^{-1} x \right)\) implies that \(\tan^{-1} x = \tfrac{\pi}{2} - \cot^{-1} x \), which proves the identity.

Continuity of Inverse Trigonometric functions

Example \(\PageIndex{1}\):

Let \(f(x)= \frac{3 \sec^{-1} (x)}{4-\tan^{-1}(x)}\). Find the values(if any) for which \(f(x)\) is continuous.

Exercise \(\PageIndex{1}\)

Let \(f(x)= \frac{3 \sec^{-1} (x)}{8+2\tan^{-1}(x)}\). Find the values(if any) for which \(f(x)\) is continuous.

Answer

Limit of Inverse Trigonometric functions

Theorem $\PageIndex{1}$

$\lim_{x \rightarrow \infty} \tan^{-1}(x) = \frac{\pi}{2} $.

$\lim_{x \rightarrow -\infty} \tan^{-1}(x) = -\frac{\pi}{2} $.

$\lim_{x \rightarrow \infty} \sec^{-1} (x)=\lim _{x \rightarrow \infty} \sec^{-1} (x)= \frac{\pi}{2} $.

Example $\PageIndex{1}$:

Find $\lim_{x \rightarrow \infty} sin\left(2\tan^{-1}(x)\right)$.

Exercise $\PageIndex{1}$

Find $\lim_{x \rightarrow -\infty} sin\left(2\tan^{-1}(x)\right)$.

Answer

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