1.8: Limits and continuity of Inverse Trigonometric functions

Inverse functions

Recall that a function \( f \) is one-to-one (often written as \( 1-1 \)) if it assigns distinct values of \( y \) to distinct values of \( x \). In other words, if \( (x_1 \neq x_2) \) then \( f(x_1) \neq f(x_2) \). Equivalently, \( f \) is one-to-one if \( f(x_1) = f(x_2) \) implies \( x_1 = x_2 \). There is a simple horizontal rule for determining whether a function \( y = f(x) \) is one-to-one: \( f \) is one-to-one if and only if every horizontal line intersects the graph of \( y = f(x) \) in the \((xy)\)-coordinate plane at most once (see Figure 5.3.3).

![Figure 5.3.3 Horizontal rule for one-to-one functions](image)

If a function \( f \) is one-to-one on its domain, then \( f \) has an inverse function, denoted by \( f^{-1} \), such that \( y = f(x) \) if and only if \( f^{-1}(y) = x \). The domain of \( f^{-1} \) is the range of \( f \).

The basic idea is that \( f^{-1} \) "undoes" what \( f \) does, and vice versa. In other words, for all \( (x) \) in the domain of \( f \), and
If \( f \) is continuous and one to one, then \( f^{-1} \) is continuous on its domain.

**Inverse Trigonometric functions**

We know from their graphs that none of the trigonometric functions are one-to-one over their entire domains. However, we can restrict those functions to *subsets* of their domains where they *are* one-to-one. For example, \( y = \sin x \) is one-to-one over the interval \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \), as we see in the graph below:

![Graph of \( y = \sin x \)](image)

For \( -\frac{\pi}{2} \le x \le \frac{\pi}{2} \) we have \( -1 \le \sin x \le 1 \), so we can define the inverse sine function \( \sin^{-1} x \) (sometimes called the arc sine and denoted by \( \arcsin x \)) whose domain is the interval \( [-1, 1] \) and whose range is the interval \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \). In other words:

\[
\begin{alignat}{3}
\sin^{-1} (\sin y) & \quad = \quad y \quad \text{for} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2} \\
\sin (\sin^{-1} x) & \quad = \quad x \quad \text{for} \quad -1 \le x \le 1
\end{alignat}
\]

**Summary of Inverse Trigonometric functions**

Let's illustrate the summary of Trigonometric functions and Inverse Trigonometric functions in following table:

<table>
<thead>
<tr>
<th>Trigonometric function</th>
<th>graph of the Trigonometric function</th>
<th>Restricted domain and</th>
<th>Inverse Trigonometric function</th>
</tr>
</thead>
</table>

UC Davis ChemWiki is licensed under a Creative Commons Attribution-Noncommercial-Share Alike 3.0 United States License.
the range

\( f(x) = \sin(x) \)

\[ \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \quad \text{and} \quad [-1,1] \]

\( f^{-1}(x) = \sin^{-1} x \)

\( f(x) = \cos(x) \)

\[ [0,\pi] \quad \text{and} \quad [-1,1] \]

\( f^{-1}(x) = \cos^{-1} x \)
\[ f(x) = \tan(x) \]

\( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \) and \( \mathbb{R} \)

\[ f(x) = \cot(x) \]
\( f(x) = \sec(x) \)

\([0, \pi]\), with \( x \neq \frac{\pi}{2} \) and \( \mathbb{R} \)

Below are examples:

Example \( \PageIndex{1} \):

Find \( \sin^{-1} \left( \sin \left( \frac{\pi}{4} \right) \right) \).

Solution:

Since \(-\frac{\pi}{2} \leq \frac{\pi}{4} \leq \frac{\pi}{2}\), we know that \( \sin^{-1} \left( \sin \left( \frac{\pi}{4} \right) \right) = \boxed{\frac{\pi}{4}} \), by Equation \ref{eqn:arcsin1}.

Example \( \PageIndex{2} \):

Find \( \sin^{-1} \left( \sin \left( \frac{5\pi}{4} \right) \right) \).

Solution:

Since \( \frac{5\pi}{4} > \frac{\pi}{2} \), we cannot use Equation \ref{eqn:arcsin1}. But we know that \( \sin \left( \frac{5\pi}{4} \right) = -\frac{1}{\sqrt{2}} \). Thus, \( \sin^{-1} \left( -\frac{1}{\sqrt{2}} \right) \) is, by definition, the angle \( y \) such that \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\) and \( \sin y = -\frac{1}{\sqrt{2}} \). That angle is
\(y = -\frac{\pi}{4}\), since

\[
\sin\left(-\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}
\]

Example \(\PageIndex{3}\):

Find \(\cos^{-1} \left(\cos\left(\frac{\pi}{3}\right)\right)\).

Solution:

Since \(0 \le \frac{\pi}{3} \le \pi\), we know that \(\cos^{-1} \left(\cos\left(\frac{\pi}{3}\right)\right) = \boxed{\frac{\pi}{3}}\), by Equation \ref{eqn:arccos1}.

Example \(\PageIndex{4}\):

Find \(\cos^{-1} \left(\cos\left(\frac{4\pi}{3}\right)\right)\).

Solution:

Since \(\frac{4\pi}{3} > \pi\), we can not use Equation \ref{eqn:arccos1}. But we know that \(\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}\). Thus, \(\cos^{-1} \left(\cos\left(\frac{4\pi}{3}\right)\right) = \frac{2\pi}{3}\) (i.e., \(120^\circ\)).

Example \(\PageIndex{5}\):

Find \(\tan^{-1} \left(\tan\left(\frac{\pi}{4}\right)\right)\).

Solution:

Since \(-\frac{\pi}{2} \le \frac{\pi}{4} \le \frac{\pi}{2}\), we know that \(\tan^{-1} \left(\tan\left(\frac{\pi}{4}\right)\right) = \boxed{\frac{\pi}{4}}\), by Equation \ref{eqn:arctan1}.

Example \(\PageIndex{6}\):

Find \(\tan^{-1} \left(\tan\left(\pi\right)\right)\).

Solution:

Since \(\pi > \frac{\pi}{2}\), we can not use Equation \ref{eqn:arctan1}. But we know that \(\tan\left(\pi\right) = 0\). Thus, \(\tan^{-1} \left(\tan\left(\pi\right)\right) = \boxed{0}\).

Example \(\PageIndex{7}\):

UC Davis ChemWiki is licensed under a Creative Commons Attribution-Noncommercial-Share Alike 3.0 United States License.
Find the exact value of \( \cos \left( \sin^{-1} \left( -\frac{1}{4} \right) \right) \).

Solution:

Let \( \theta = \sin^{-1} \left( -\frac{1}{4} \right) \). We know that \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \), so since \( \sin \theta = -\frac{1}{4} < 0 \), \( \theta \) must be in QIV. Hence \( \cos \theta > 0 \). Thus,

\[
\cos^2 \theta = 1 - \sin^2 \theta = 1 - \left( -\frac{1}{4} \right)^2 = \frac{15}{16}
\]

\[
\Rightarrow \cos \theta = \frac{\sqrt{15}}{4}.
\]

Note that we took the positive square root above since \( \cos \theta > 0 \). Thus, \( \cos \left( \sin^{-1} \left( -\frac{1}{4} \right) \right) = \boxed{\frac{\sqrt{15}}{4}} \).

Example \( \PageIndex{8} \):

Show that \( \tan \left( \sin^{-1} x \right) = \frac{x}{\sqrt{1 - x^2}} \) for \( -1 < x < 1 \).

Solution:

When \( x=0 \), the Equation holds trivially, since

\[
\tan \left( \sin^{-1} 0 \right) = \tan 0 = 0 = \frac{0}{\sqrt{1 - 0^2}}.
\]

Now suppose that \( 0 < x < 1 \). Let \( \theta = \sin^{-1} x \). Then \( \theta \) is in QI and \( \sin \theta = x \). Draw a right triangle with an angle \( \theta \) such that the opposite leg has length \( x \) and the hypotenuse has length \( 1 \), as in Figure 5.3.10 (note that this is possible since \( 0 < x < 1 \)). Then \( \sin \theta = \frac{x}{1} = x \). By the Pythagorean Theorem, the adjacent leg has length \( \sqrt{1 - x^2} \). Thus, \( \tan \theta = \frac{x}{\sqrt{1 - x^2}} \).

If \( -1 < x < 0 \) then \( \theta = \sin^{-1} x \) is in QIV. So we can draw the same triangle except that it would be "upside down" and we would again have \( \tan \theta = \frac{x}{\sqrt{1 - x^2}} \), since the tangent and sine have the same sign (negative) in QIV. Thus, \( \tan \left( \sin^{-1} x \right) = \frac{x}{\sqrt{1 - x^2}} \) for \( -1 < x < 1 \).

UC Davis ChemWiki is licensed under a Creative Commons Attribution-Noncommercial-Share Alike 3.0 United States License.
Example (PageIndex{9}): 

Prove the identity \( \tan^{-1} x = \cot^{-1} x \).

Solution:

Let \( \theta = \cot^{-1} x \). Using relations, we have

\[
\tan \left( \frac{\pi}{2} - \theta \right) = -\tan \left( \theta - \frac{\pi}{2} \right) = \cot \theta = \cot (\cot^{-1} x) = x \tag{2}
\]

by Equation (eqn:arccot2). So since \( \tan(\tan^{-1} x) = x \) for all \( x \), this means that \( \tan(\tan^{-1} x) = \tan \left( \frac{\pi}{2} - \theta \right) \). Thus, \( \tan(\tan^{-1} x) = \tan \left( \frac{\pi}{2} - \theta \right) \). Now, we know that \( 0 < \cot^{-1} x < \pi \), so \( -\tan \left( \frac{\pi}{2} - \theta \right) \) is in the restricted subset on which the tangent function is one-to-one. Hence, \( \tan(\tan^{-1} x) = \tan \left( \frac{\pi}{2} - \theta \right) \) implies that \( \tan^{-1} x = \frac{\pi}{2} - \cot^{-1} x \), which proves the identity.

---

**Continuity of Inverse Trigonometric functions**

Example (PageIndex{1}): 

Let \( f(x) = \frac{3 \sec^{-1}(x)}{4 - \tan^{-1}(x)} \). Find the values (if any) for which \( f(x) \) is continuous.

Exercise (PageIndex{1})

Let \( f(x) = \frac{3 \sec^{-1}(x)}{8 + 2\tan^{-1}(x)} \). Find the values (if any) for which \( f(x) \) is continuous.

**Answer**
Limit of Inverse Trigonometric functions

Theorem \(\PageIndex{1}\)

\[ \lim_{x \rightarrow \infty} \tan^{-1}( x) = \frac{\pi}{2}. \]

\[ \lim_{x \rightarrow -\infty} \tan^{-1}( x) = -\frac{\pi}{2}. \]

\[ \lim_{x \rightarrow \infty} \sec^{-1}( x) = \lim_{x \rightarrow -\infty} \sec^{-1}( x) = \frac{\pi}{2}. \]

Example \(\PageIndex{1}\):

Find \( \lim_{x \rightarrow \infty} \sin\left( 2\tan^{-1}( x)\right) \).

Exercise \(\PageIndex{1}\)

Find \( \lim_{x \rightarrow -\infty} \sin\left( 2\tan^{-1}( x)\right) \).

Answer

Contributors

- Michael Corral (Schoolcraft College). The content of this page is distributed under the terms of the GNU Free Documentation License, Version 1.2.

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

- Pamini Thangarajah (Mount Royal University, Calgary, Alberta, Canada)