1.8: Limits and continuity of Inverse Trigonometric functions

Inverse functions

Recall that a function \( f \) is \textbf{one-to-one} (often written as \( 1-1 \)) if it assigns distinct values of \( y \) to distinct values of \( x \). In other words, if \( (x_1 \neq x_2) \) then \( f(x_1) \neq f(x_2) \). Equivalently, \( f \) is one-to-one if \( f(x_1) = f(x_2) \) implies \( x_1 = x_2 \). There is a simple horizontal rule for determining whether a function \( y = f(x) \) is one-to-one: \( f \) is one-to-one if and only if every horizontal line intersects the graph of \( y = f(x) \) in the \( xy \)-coordinate plane at most once (see Figure 5.3.3).

![Horizontal rule for one-to-one functions](image)

\( f \) is one-to-one \hspace{1cm} \( f \) is not one-to-one

\textbf{Figure 5.3.3} Horizontal rule for one-to-one functions

If a function \( f \) is one-to-one on its domain, then \( f \) has an \textbf{inverse function}, denoted by \( f^{-1} \), such that \( y = f(x) \) if and only if \( f^{-1}(y) = x \). The domain of \( f^{-1} \) is the range of \( f \).

The basic idea is that \( f^{-1} \) "undoes" what \( f \) does, and vice versa. In other words, for all \( x \) in the domain of \( f \), and \( y \) in the range of \( f \),

\[ f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y. \]
Theorem \(\PageIndex{1}\)

If \(f\) is continuous and one to one, then \(f^{-1}\) is continuous on its domain.

### Inverse Trigonometric functions

We know from their graphs that none of the trigonometric functions are one-to-one over their entire domains. However, we can restrict those functions to subsets of their domains where they are one-to-one. For example, \(y = \sin x\) is one-to-one over the interval \([-\frac{\pi}{2}, \frac{\pi}{2}]\), as we see in the graph below:

![Graph of \(y = \sin x\)](graph.png)

For \(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\) we have \(-1 \le \sin x \le 1\), so we can define the inverse sine function \(y = \sin^{-1} x\) (sometimes called the arc sine and denoted by \(y = \arcsin(x)\)) whose domain is the interval \([-1, 1]\) and whose range is the interval \([-\frac{\pi}{2}, \frac{\pi}{2}]\). In other words:

\[
\begin{alignat*}{3}
f(f^{-1}(y)) &= y \quad \text{for all } y \text{ in the range of } f.
\end{alignat*}
\]

### Summary of Inverse Trigonometric functions

Let's illustrate the summary of Trigonometric functions and Inverse Trigonometric functions in following table:

<table>
<thead>
<tr>
<th>Trigonometric function</th>
<th>graph of the Trigonometric function</th>
<th>Restricted domain and</th>
<th>Inverse Trigonometric function</th>
</tr>
</thead>
</table>

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the range

\[ f(x) = \sin(x) \]
\[ \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \]
\[ f^{-1}(x) = \sin^{-1} x \]
and \[ [-1,1] \]

\[ f(x) = \cos(x) \]
\[ [0, \pi] \]
\[ f^{-1}(x) = \cos^{-1} x \]
and \[ [-1,1] \]
\[ f(x) = \tan(x) \quad \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \cup \mathbb{R} \]

\[ f^{-1}(x) = \tan^{-1} x \]

\[ f(x) = \cot(x) \]
Below are examples:

Example \(\PageIndex{1}\):

Find \(\sin^{-1} \left(\sin\left(\frac{\pi}{4}\right)\right)\).

Solution:

Since \(-\frac{\pi}{2} \leq \frac{\pi}{4} \leq \frac{\pi}{2}\), we know that \(\sin^{-1} \left(\sin\left(\frac{\pi}{4}\right)\right) = \boxed{\frac{\pi}{4}}\), by Equation \ref{eqn:arcsin1}.

Example \(\PageIndex{2}\):

Find \(\sin^{-1} \left(\sin\left(\frac{5\pi}{4}\right)\right)\).

Solution:

Since \(\frac{5\pi}{4} > \frac{\pi}{2}\), we cannot use Equation \ref{eqn:arcsin1}. But we know that \(\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}\). Thus, \(\sin^{-1} \left(\sin\left(\frac{5\pi}{4}\right)\right) = \sin^{-1} \left(-\frac{\sqrt{2}}{2}\right)\) is, by definition, the angle \(y\) such that \(-\frac{\sqrt{2}}{2} \leq y \leq \frac{\pi}{2}\) and \(\sin y = -\frac{\sqrt{2}}{2}\). That angle is
\( y = -\frac{\pi}{4} \), since

\[
\sin\left( -\frac{\pi}{4} \right) = -\sin\left( \frac{\pi}{4} \right) = -\frac{1}{\sqrt{2}}.
\]

Example \((\PageIndex{3})\):

Find \( \cos^{-1} \left( \cos\left( \frac{\pi}{3} \right) \right) \).

**Solution:**

Since \( 0 \leq \frac{\pi}{3} \leq \pi \), we know that \( \cos^{-1} \left( \cos\left( \frac{\pi}{3} \right) \right) = \boxed{\frac{\pi}{3}} \), by Equation \ref{eqn:arccos1}.

Example \((\PageIndex{4})\):

Find \( \cos^{-1} \left( \cos\left( \frac{4\pi}{3} \right) \right) \).

**Solution:**

Since \( \frac{4\pi}{3} > \pi \), we can not use Equation \ref{eqn:arccos1}. But we know that \( \cos\left( \frac{4\pi}{3} \right) = -\frac{1}{2} \). Thus, \( \cos^{-1} \left( \cos\left( \frac{4\pi}{3} \right) \right) \) is, by definition, the angle \( y \) such that \( 0 \leq y \leq \pi \) (i.e. \( 120^{\circ} \)).

Thus, \( \cos^{-1} \left( \cos\left( \frac{4\pi}{3} \right) \right) = \boxed{\frac{2\pi}{3}} \).

Example \((\PageIndex{5})\):

Find \( \tan^{-1} \left( \tan\left( \frac{\pi}{4} \right) \right) \).

**Solution:**

Since \( -\frac{\pi}{2} \leq \frac{\pi}{4} \leq \frac{\pi}{2} \), we know that \( \tan^{-1} \left( \tan\left( \frac{\pi}{4} \right) \right) = \boxed{\frac{\pi}{4}} \), by Equation \ref{eqn:arctan1}.

Example \((\PageIndex{6})\):

Find \( \tan^{-1} \left( \tan\left( \pi \right) \right) \).

**Solution:**

Since \( \pi > \frac{\pi}{2} \), we can not use Equation \ref{eqn:arctan1}. But we know that \( \tan\left( \pi \right) = 0 \). Thus, \( \tan^{-1} \left( \tan\left( \pi \right) \right) = \boxed{0} \).

Example \((\PageIndex{7})\):
Find the exact value of $\cos\left(\sin^{-1}\left(-\frac{1}{4}\right)\right)$.

**Solution:**

Let $\theta = \sin^{-1}\left(-\frac{1}{4}\right)$. We know that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, so since $\sin\theta = -\frac{1}{4} < 0$, $\theta$ must be in QIV. Hence $\cos\theta > 0$. Thus,

$$\cos^2 \theta = 1 - \sin^2 \theta = 1 - \left(-\frac{1}{4}\right)^2 = \frac{15}{16} \Rightarrow \cos \theta = \frac{\sqrt{15}}{4}.$$

Note that we took the positive square root above since $\cos\theta > 0$. Thus, $\cos\left(\sin^{-1}\left(-\frac{1}{4}\right)\right) = \boxed{\frac{\sqrt{15}}{4}}$.

Example: Show that $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1 - x^2}}$ for $-1 < x < 1$.

**Solution:**

When $x = 0$, the Equation holds trivially, since

$$\tan(\sin^{-1} 0) = \tan 0 = 0 = \frac{0}{\sqrt{1 - 0^2}}.$$

Now suppose that $0 < x < 1$. Let $\theta = \sin^{-1} x$. Then $\theta$ is in QI and $\sin\theta = x$. Draw a right triangle with an angle $\theta$ such that the opposite leg has length $x$ and the hypotenuse has length $1$, as in Figure 5.3.10 (note that this is possible since $0 < x < 1$). Then $\sin\theta = \frac{x}{1} = x$. By the Pythagorean Theorem, the adjacent leg has length $\sqrt{1 - x^2}$. Thus, $\tan\theta = \frac{x}{\sqrt{1 - x^2}}$.

If $-1 < x < 0$, then $\sin x$ is in QIV. So we can draw the same triangle except that it would be "upside down" and we would again have $\tan\theta = \frac{x}{\sqrt{1 - x^2}}$, since the tangent and sine have the same sign (negative) in QIV. Thus, $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1 - x^2}}$ for $-1 < x < 1$. 
Example $(\PageIndex{9})$:

Prove the identity $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$.

**Solution:**

Let $\theta = \cot^{-1} x$. Using relations, we have
\[
\tan\left( \frac{\pi}{2} - \theta \right) = -\tan\left( \theta - \frac{\pi}{2} \right) = \cot \theta = \cot(\cot^{-1} x) = x,
\]
by Equation $(\ref{eqn:arccot2})$. So since $\tan(\tan^{-1} x) = x$ for all $x$, this means that $\tan(\tan^{-1} x) = \tan\left( \frac{\pi}{2} - \theta \right)$. Thus, $\tan(\tan^{-1} x) = \tan\left( \frac{\pi}{2} - \cot^{-1} x \right)$. Now, we know that $0 < \cot^{-1} x < \pi$, so $\frac{\pi}{2} - \cot^{-1} x$ is in the restricted subset on which the tangent function is one-to-one. Hence, $\tan(\tan^{-1} x) = \tan\left( \frac{\pi}{2} - \cot^{-1} x \right)$ implies that $\tan^{-1} x = \frac{\pi}{2} - \cot^{-1} x$, which proves the identity.

**Continuity of Inverse Trigonometric functions**

Example $(\PageIndex{1})$:

Let $f(x) = \frac{3 \sec^{-1} (x)}{4 - \tan^{-1}(x)}$. Find the values (if any) for which $f(x)$ is continuous.

Exercise $(\PageIndex{1})$

Let $f(x) = \frac{3 \sec^{-1} (x)}{8 + 2\tan^{-1}(x)}$. Find the values (if any) for which $f(x)$ is continuous.

**Answer**
Limit of Inverse Trigonometric functions

Theorem \(\PageIndex{1}\)

\[
\lim_{{x \to \infty}} \tan^{-1}(x) = \frac{\pi}{2}.
\]

\[
\lim_{{x \to -\infty}} \tan^{-1}(x) = -\frac{\pi}{2}.
\]

\[
\lim_{{x \to \infty}} \sec^{-1}(x) = \lim_{{x \to -\infty}} \sec^{-1}(x) = \frac{\pi}{2}.
\]

Example \(\PageIndex{1}\):

Find \(\lim_{{x \to \infty}} \sin(2\tan^{-1}(x))\).

Exercise \(\PageIndex{1}\)

Find \(\lim_{{x \to -\infty}} \sin(2\tan^{-1}(x))\).

Answer

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