Nonsymmetric Macdonald Polynomials

Background

Since we will be restricting ourselves to $GL_n$, we consider the weight lattice $\langle X = \mathbb{Z}^n \rangle$, with simple roots $\langle \alpha_i = e_i - e_{i+1} \rangle$, where $\langle e_i \rangle$ is the $i$-th unit vector. We identify the co-weight lattice $\langle X^\vee \rangle$ with $\langle X \rangle$ using the standard inner product on $\langle \mathbb{Z}^n \rangle$ so that $\langle \alpha_i \rangle = \langle e_i \rangle$. Therefore the dominant weights are $\langle \lambda \rangle \geq \langle e_i \rangle$ for all $i$ and are partitions.

The affine weight lattice is $\langle \widehat{X} = X \oplus \mathbb{Z} \delta \rangle$ where $\langle \delta \rangle$ is the smallest positive imaginary root, or null root. The extra simple root is $\langle \alpha_0 = \delta - \theta \rangle$ where $\langle \theta = e_1 - e_n \rangle$ is the highest root of $\langle GL_n \rangle$. The extra affine roots are $\langle \widehat{R}_+ = \{ e_i - e_j + k \delta \mid i \neq j, k > 0, \text{ and if } i > j, k > 0 \} \rangle$. We denote $\langle x^\delta = q \rangle$, so $\langle x^\alpha \rangle = x_i / x_{i+1}$ and $\langle x^\alpha_0 \rangle = q x_1 / x_n$. Therefore the group ring $\langle \mathbb{Q}(q,t) \widehat{X} \rangle$ by extending scalars.

The inner product we want is Cherednik's inner product on $\langle \mathbb{Q}(q,t) \rangle$ given by $\langle \langle f, g \rangle_{q,t} = [x^0](f \bar{g} \Delta_1) \rangle$ where $\bar{\cdot}$ is the involution given by $\bar{q} = q^{-1}, \bar{t} = t^{-1}, \bar{x}_i = x_i^{-1}$, and $\Delta_1 = \Delta / ([x^0](\Delta))$ with $\langle \Delta = \prod_{\alpha \in \widehat{R}_+} \frac{1 - x^\alpha}{1 - t x^\alpha} \rangle$. Here $\Delta$ denotes the coefficient of $\lambda$ in $\langle f \bar{g} \rangle$. It is known that $\langle \lambda \rangle \Delta \in \mathbb{Q}(q,t)$ and $\langle x^\lambda \rangle \Delta = \overline{\Delta}$. Hence $\langle \lambda \rangle = \langle \bar{\lambda} \rangle$ and $\langle \lambda \rangle = \langle \bar{\lambda} \rangle$.

The Bruhat order on $\langle X \rangle$ is given by identification with $\langle \widehat{W} / W_0 \rangle$ where $\langle W_0 = S_n \rangle$ is the Weyl group of...
\((\text{GL}_n)\) and \(\langle \text{affine } W = W_0 \ltimes X \rangle\) is the extended affine Weyl group, equipped with the usual Bruhat order. Explicitly for \((\text{GL}_n)\) we have \(\lambda > \sigma_{ij}(\lambda)\) if \(i < j\) and \(\sigma_{ij}(\lambda)\) is the transposition \((i \; j)\). If \(\langle \lambda_i < \lambda_j \rangle\) and \(\langle \lambda_j - \lambda_i > 1 \rangle\), then \(\langle \sigma_{ij}(\lambda) > \lambda \rangle\).

We define the nonsymmetric Macdonald polynomials \((E_{\mu}(x; q, t))\) for \((\mu \in X)\) are uniquely characterized by the conditions:

1. Triangularity: \((E_{\mu}(x) \in x^\lambda + \text{mathbb}(Q)(q, t)\{ x^\lambda \mid \lambda \in \text{affine } X \text{ and } \lambda_i < \lambda_j \})\)
2. Orthogonality: \((\langle E_{\mu}(x), E_{\nu}(x) \rangle = 0\) for \((\mu \neq \nu)\))

One last note, the notation used here might differ from that used elsewhere.

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**Hecke Algebras**

The affine Hecke algebra \((\mathcal{H})\) is the \(\mathbb{Q}(t)\)-algebra \(\langle T_0, T_1, \ldots, T_{n-1} \rangle\) which satisfy the braid relations \(T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}\), \(T_i T_j = T_j T_i\) \((i - j \neq \pm 1)\), and the quadratic relation \((T_i - t)(T_i + 1) = 0\). The (unextended) affine Weyl group \((W = \langle s_0, s_1, \ldots s_{n-1} \rangle\) which satisfy the braid relations above and act naturally on \((\text{affine } X)\), as well as the extensions, by \(s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i\).

Explicitly for \((i \neq 0)\), these are the usual transpositions, and \((s_0 f(x_1, \ldots, x_n) = f(q x_n, x_2, \ldots, x_{n-1}, x_1/q)\).

**Cherednik’s representation** of \((\mathcal{H})\) is given by the formula \(T_i x^\lambda = t x^{s_i(\lambda)} + (t - 1) x^{s_i(\lambda)}\) \((i \neq 0)\) which satisfies the braid relations above and act naturally on \((\text{affine } X)\), as well as the extensions, by \(s_i (\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i\).

**Nonsymmetric Macdonald Polynomials**

From Bernstein’s relations in \((\mathcal{H})\), the simultaneous eigenfunctions \((E_{\mu}(x; q, t))\) satisfy the relations \((E_{s_i(\mu)}(x; q, t))\) \((i \neq 0)\) \((\mu_i > \mu_{i+1})\) and where \((w_{\mu}(\rho))\) is the maximal length permutation such that \((w_{\mu}(\rho))\) is dominant. Next we need the following automorphisms \((\pi_{\lambda}(\lambda_1, \ldots, \lambda_n) = (\lambda_n + 1, \lambda_1, \ldots, \lambda_{n-1})\) \((\Psi f(x_1, \ldots, x_n) = x_1 f(x_2, \ldots, x_n, q^{-1} x_1)\). We can show that \((E_{\pi}(\mu)(x; q, t) = q^{\mu_n} \Psi E_{\mu}(x; q, t))\) and these two relations are known as the Knop-Sahi recurrence.

Next we can show that the second of Knop-Sahi recurrences, the special case of the first recurrence in which \((\mu_0 = 0)\), \((\mu_{i+1} = 0)\) completely characterize the nonsymmetric Macdonald polynomials. From the fact that they are
eigenfunctions, we get that nonsymmetric Macdonald polynomials exist.

**Combinatorics**

We begin by considering diagrams for the weak compositions \( \mu \in \mathbb{Z}_{\geq 0}^n \) of length \( n \) by drawing the \( (i) \)-th column as a length \( \mu_i \) column aligned at the bottom. We denote this by \( \text{dg}(\mu) \). We also consider augmented diagrams \( \text{dg}^*(\mu) \) which is the usual diagram but adding a row of length \( \mu_n \) to the base.

We define the following sets of a cell \( (i, j) \in \text{dg}(\mu) \):

- The leg is the set of cells directly above \( (i, j) \). So all cells \( (i', j') \in \text{dg}(\mu) \) such that \( i' < i \).
- The left arm is the set of cells to the left of \( (i, j) \) of equal or smaller height columns. So all cells \( (i', j) \in \text{dg}(\mu) \) such that \( i' < i \) and \( \mu_i = \mu_{i'} \).
- The right arm is the set of cells to the right of \( (i, j) \) in the row below \( (i, j) \) of strictly smaller height columns. So all cells \( (i', j+1) \) such that \( i' < i \) and \( \mu_i = \mu_{i'} \).
- The arm is the union of the left and right arms.

We now define the set of statistics on \( (i, j) \). We begin by defining \( l(u) = \lvert \text{leg}(u) \rvert = \mu_i - j \) and \( a(u) = \lvert \text{arm}(u) \rvert \). Using these statistics, if \( \mu_i > \mu_{i+1} \), we can reformulate our recursion by

\[
\mathcal{E}_{s_i(\mu)}(x; q, t) = \left( T_i + \frac{1 - t}{1 - q^{l(u)+1} t^{a(u)}} \right) \mathcal{E}_{\mu}(x; q, t),
\]

where \( u = (i, \mu_{i+1} + 1) \). We can also define an integral form for the non-symmetric Macdonald polynomials by

\[
\mathcal{E}_{\mu}(x; q, t) = \prod_{u \in \text{dg}(\mu)} \left( 1 - q^{l(u)+1} t^{a(u)+1} \right) \mathcal{E}_{\mu}(x; q, t).
\]

Our next statistics will be defined on fillings of the diagrams, which are just maps \( \sigma : \text{dg}(\mu) \to [n] \). We can augment the filling by defining the map \( \text{dg}^*(\mu) \) with \( \text{dg}^*(\mu) \) and agrees with \( \sigma \) everywhere else. We say two cells \( (a,b), (i,j) \) attack each other if

- they are in the same row, i.e. \( b = j \), or
- they are in consecutive rows and the box in the lower row is to the right of the one in the upper row, i.e. \( i < a \) and \( b = j - 1 \).

We say an augmented filling is non-attacking if \( \sigma \neq \text{dg}^*(\mu) \) for all attacking pairs \( (u,v) \) and \( \text{dg}^*(\mu) \). We say a filling is non-attacking if its augmented filling is non-attacking.

Next let \( d(u) = (i, j-1) \) be the box directly below \( u = (i, j) \). A descent in the filling is a box \( d(u) \in \text{dg}^*(\mu) \) such that \( (d(u)) \in \text{dg}^*(\mu) \) and \( \text{sgn}(\dagger) \). We denote by \( \text{Des}(\sigma) \) as the set of descents of \( \sigma \) and the major index is \( \sum_{u \in \text{Des}(\sigma)} (l(u) + 1) \). We define the reading order of boxes in a diagram by reading row by row from right-to-left, top-to-bottom, i.e. \( (i, j) < (a, b) \) if \( j > b \), or if \( j = b \) and \( i > a \). An inversion of a filling \( \text{Inv}(\sigma) \) is a pair of attacking boxes \( (u, v) \) such that \( u < v \) in the reading order and \( \text{Inv}(\sigma) \) and \( \text{Inv}(\sigma) \). We denote this set by \( \text{In} \) and the inversion statistic by \( \text{inv}(\sigma) = \sum_{u \in \text{Inv}(\sigma)} (l(u) + 1) \).
We also define the co-inversion statistic by $\text{coinv}(\widehat{\sigma}) = -\text{inv}(\widehat{\sigma}) + \sum_{u \in \text{dg}(\mu)} a(u)$. Thus we can compute the non-symmetric Macdonald polynomials by $E_{\mu}(x; q, t) = \sum_{\sigma} x^{\sigma} q^{\text{maj}(\widehat{\sigma})} t^{\text{coinv}(\widehat{\sigma})} \prod_{u \in \text{dg}(\mu)} \frac{1 - t}{1 - q^{l(u)+1} t^{a(u)+1}}$, where we sum over all non-attacking fillings of $\text{dg}(\mu)$ and $x^{\sigma} = \prod_{u \in \text{dg}(\mu)} x_{\sigma(u)}$. We also have the integral form as $\mathcal{E}_{\mu}(x; q, t) = \sum_{\sigma} x^{\sigma} q^{\text{maj}(\widehat{\sigma})} t^{\text{coinv}(\widehat{\sigma})} \prod_{u \in \text{dg}(\mu)} (1 - q^{l(u)+1} t^{a(u)+1}) \prod_{u \in \text{dg}(\mu)} 1 - q^{l(u)+1} t^{a(u)+1} (1 - t)$.