Macdonald Polynomials and Demazure Characters

Introduction

We will here discuss the connection between nonsymmetric Macdonald polynomials and the characters of Demazure modules for $\widehat{\mathfrak{sl}}(n)$ as given in [3]. We assume a familiarity with affine (untwisted) Lie algebras, specifically $\widehat{\mathfrak{sl}}(n)$, but will give all necessary facts about Demazure modules and nonsymmetric Macdonald polynomials here.

Nonsymmetric Macdonald Polynomials

Recall that nonsymmetric Macdonald polynomials $E_\lambda(z_1, \ldots, z_n, q, t)$ are indexed by compositions $\lambda \in \mathbb{N}^n$ and that they form a basis of $\mathbb{C}(q,t)[z_1, \ldots, z_n]$. Henceforth we specialize to $t = 0$, and write

$$E_\lambda = E_\lambda(z_1, \ldots, z_n, q, 0).$$

We can generate these polynomials recursively via the endomorphisms $\Phi, H_0, H_1, \ldots, H_{n-1}$ acting on the space $\mathbb{Z}(q,q^{-1})[z_1, \ldots, z_n]$ (note that when we specialize to $t = 0$) we drop from the space $\mathbb{C}(q,t)[z_1, \ldots, z_n]$ to $\mathbb{Z}(q,q^{-1})[z_1, \ldots, z_n]$). $\Phi, H_0, H_1, \ldots, H_{n-1}$, are defined such that

$$\bar{H}_i = s_i - z_{i+1}\frac{1 - s_i}{z_i - z_{i+1}} \quad 1 \leq i \leq n-1,$$

$$\Phi f(z_1, \ldots, z_n) = z_n f^{-1}(q^{-1}z_n, z_1, \ldots, z_{n-1}).$$

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There is an \( \bar{H}_0 \) too but we will not discuss it. The recursive rules tell us that after setting \( E_{(0^n)} = 1 \), then
\[
q^\lambda \Phi E_{(\lambda_1, \ldots, \lambda_n)} = E_{(\lambda_2, \ldots, \lambda_n, \lambda_1 + 1)} \]
\[
q^{\lambda_1 - \lambda_n + 1} \bar{H}_0 E_\lambda = E_{(\lambda_n - 1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_1 + 1)} \quad \text{if } \lambda_1 > \lambda_n - 1
\]
and otherwise \( q^{\lambda_1 - \lambda_n + 1} \bar{H}_0 E_\lambda = E_\lambda \). Finally,
\[
\bar{H}_i E_\lambda = E_{s_i \lambda} \quad \text{if } \lambda_i < \lambda_{i+1}
\]
and otherwise \( \bar{H}_i E_\lambda = E_\lambda \). These operators should be reminiscent of the action of the Weyl group of \( \widehat{\mathfrak{sl}}(n) \) on compositions.

As an example suppose that for \( n = 3 \) we want to generate \( E_{(1,2,1)} \). Then we could apply the composition \( \bar{H}_2 \Phi^4 \) to \( E_{(0,0,0)} \) to get
\[
\Phi(E_{(0,0,0)}) = E_{(0,0,1)} = z_3, \\
\Phi(E_{(0,0,1)}) = E_{(0,1,1)} = z_2z_3, \\
\Phi(E_{(0,1,1)}) = E_{(1,1,1)} = z_1z_2z_3, \\
\Phi(E_{(1,1,1)}) = E_{(1,1,2)} = z_1z_2z_3^2, \\
\bar{H}_2(E_{(1,1,2)}) = E_{(1,2,1)} = z_1z_2^2z_3 + z_1z_2z_3^2
\]

**Demazure Modules**

In this section we let \( \mathfrak{g} \) be a Kac-Moody algebra associated with Cartan datum \( (\mathfrak{h}, \Pi, \Pi^\vee, P, P^\vee) \). We closely follow chapter 2 of [1]. Recall that a \( \mathfrak{g} \)-module \( V \) is a weight module if it admits a weight space decomposition:
\[
V = \bigoplus_{\mu} V_\mu
\]
where
\[
V_\mu = \{ v \in V \mid hv = \mu(h)v \text{ for all } h \in \mathfrak{h} \}
\]
A vector \( v \in V_\mu \) is called a weight vector of weight \( \mu \) if \( e_i v = 0 \) for all \( i \in I \), \( \langle \mu \rangle \) is called a maximal weight vector. The dimension \( \dim V_\mu \) is called the weight multiplicity of \( \mu \). When \( \dim V_\mu < \infty \) for all \( \mu \), the

**character** of \( V \) is defined to be
\[ V = \sum \dim V_\mu e^\mu \]

where \( (e^\mu) \) are formal basis elements of the group algebra \( F[\mathfrak{h}^*] \) with multiplication \( e^\lambda e^\mu = e^{\lambda + \mu} \). We call a \( \mathfrak{g} \) module \( V \) a **highest weight module** of **highest weight** \( \lambda \) if there exists a nonzero vector \( v_\lambda \in V \) such that

\[
\begin{align*}
\langle e_i v_\lambda \rangle &= 0 \; \; \text{for all } i \\
\langle h v_\lambda \rangle &= \lambda(h) v \\
\langle V = U(\mathfrak{g}) v_\lambda \rangle &= \text{or } U^- v_\lambda = V 
\end{align*}
\]

where we here use the decomposition \( U(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+ \) of the universal enveloping algebra of \( \mathfrak{g} \). An element \( \Lambda \in \mathfrak{h}^* \) is a dominant integral weight if \( \Lambda \) belongs to the set,

\[ P^+ = \{ \Lambda \in P | \lambda(h_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i \} \]

The irreducible highest weight \( \mathfrak{g} \)-modules \( V(\Lambda) \) where \( \Lambda \) is a dominant integral weight have the special property that the Chevalley generators \( e_i \) and \( f_i \) are locally nilpotent on \( V(\Lambda) \). This allows us to construct a well-defined automorphism

\[ \tau_i = (\exp f_i)(\exp (-e_i))(\exp f_i) \]

where the action of \( \tau_i \) on weight spaces is given by

\[ \tau_i V_\lambda = V_{\lambda + \alpha_i} \; \; \text{for all } i \in I, \lambda \in \text{wt}(V) \]

If we still assume that \( \Lambda \) is a dominant integral weight, \( V = V(\Lambda) \) the unique irreducible highest weight \( \widehat{\mathfrak{sl}}(n) \)-module with highest weight \( \lambda \), then the weight space \( V_\lambda \) of \( V \) with weight \( \lambda \), generates a \( U^+ \)-module \( E_\lambda \) which is called a **Demazure module**. Note that Demazure modules are finite dimensional, and also that they form a filtration on \( V(\Lambda) \) which is compatible with the Bruhat order on \( W \):

\[ w \leq w' \implies E_w(\Lambda) \subseteq E_{w'}(\Lambda) \]

We can also define **Demazure operators** that act on the group ring of the weight lattice \( P \):

\[ \Delta_i = \frac{1 - e^{-\alpha_i} s_i}{1 - e^{-\alpha_i}} \]

where \( s_i \) is the simple reflection in the Weyl group with respect to simple root \( \alpha_i \). To \( w \in W \) with reduced decomposition \( w = s_{i_1} s_{i_2} \ldots s_{i_{\ell(w)}} \) we can then associate the Demazure operator
There is a nice connection between characters and Demazure operators given by the formula [2]:
\[ \chi(E_w(\Lambda)) = \Delta_w(e^\Lambda). \]

**The Connection**

We let \( \Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1} \) be the \((n)\)-fundamental weights of \( \widehat{\mathfrak{sl}}(n) \). Recall that these \( \Lambda_i \) are defined such that \((\Lambda_i, \alpha_j) = \delta_{ij}\). Finally,
\[ \delta = \sum^{n-1}_{i = 0} \alpha_i. \]

For the connection between \( E_\lambda \) and characters of Demazure modules we want to relate the action of \( \bar{H}_i \) and \( \Phi \) to operators on \( P \). More specifically, we would like a commutative diagram

We can get this by defining \( \pi : \mathbb{Z}[q,q^{-1}][z_1, \ldots, z_n] \rightarrow P \) on generators by
\[ \pi(z_i) = e^{\Lambda_{i-1} - \Lambda_i}, \quad \pi(z_n) = e^{\Lambda_{n-1} - \Lambda_0}, \quad \pi(q) = e^{-\sum^{n-1}_{i = 0} \alpha_i}. \]

(note that this definition is slightly different to that found in the paper). We get a similar commutative diagram for \( \Phi \):
The main result of [3] is then that through the homomorphism $\pi$, we can identify

$$[q^{u(\lambda) + u(\eta_{|\lambda|})} E_\lambda; \chi(E_w(\Lambda_i))]$$

where $u(\lambda)$ and $\eta_{\lambda}$ (this is a partition) depend only on $\lambda$ and $i = |\lambda| \text{mod} \; n$ and where $w$ is a specific affine Weyl group element defined such that $w$ acts on $\eta_{|\lambda|}$ to give $\lambda$.

References

