5.7: Integrals Resulting in Inverse Trigonometric Functions and Related Integration Techniques

In this section we focus on integrals that result in inverse trigonometric functions. We have worked with these functions before. Recall, that trigonometric functions are not one-to-one unless the domains are restricted. When working with inverses of trigonometric functions, we always need to be careful to take these restrictions into account. Also, we previously developed formulas for derivatives of inverse trigonometric functions. The formulas developed there give rise directly to integration formulas involving inverse trigonometric functions.

Integrals that Result in Inverse Trigonometric Functions

Let us begin this last section of the chapter with the three formulas. Along with these formulas, we use substitution to evaluate the integrals. We prove the formula for the inverse sine integral.

Rule: Integration Formulas Resulting in Inverse Trigonometric Functions

The following integration formulas yield inverse trigonometric functions:

\[
\begin{align}
∫ \frac{du}{\sqrt{a^2−u^2}} &= \arcsin \left(\frac{u}{a}\right) + C \\
∫ \frac{du}{a^2+u^2} &= \frac{1}{a} \arctan \left(\frac{u}{a}\right) + C \\
∫ \frac{du}{u \sqrt{u^2−a^2}} &= \frac{1}{a} \text{arcsec} \left(\frac{|u|}{a}\right) + C
\end{align}
\]

Proof of the first formula

Let \( y=\arcsin \frac{x}{a} \). Then \( a \sin y=x \). Now using implicit differentiation, we obtain
\[ \frac{d}{dx}(a \sin y) = \frac{d}{dx}(x) \]
\[ a \cos y \frac{dy}{dx} = 1 \]
\[ \frac{dy}{dx} = \frac{1}{a \cos y}. \]

For \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \cos y \geq 0\). Thus, applying the Pythagorean identity \((\sin^2 y + \cos^2 y = 1)\), we have \(\cos y = \sqrt{1 - \sin^2 y}\). This gives
\[
\begin{align}
\frac{1}{a \cos y} &= \frac{1}{a \sqrt{1 - \sin^2 y}} \\
&= \frac{1}{\sqrt{a^2 - a^2 \sin^2 y}} \\
&= \frac{1}{\sqrt{a^2 - x^2}}.
\end{align}
\]

Then for \(-a \leq x \leq a\) we have
\[
\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \left( \frac{u}{a} \right) + C.
\]

Example \(\PageIndex{1}\): Evaluating a Definite Integral Using Inverse Trigonometric Functions

Evaluate the definite integral
\[
\int_{0}^{1/2} \frac{dx}{\sqrt{1-x^2}}. \nonumber
\]

Solution

We can go directly to the formula for the antiderivative in the rule on integration formulas resulting in inverse trigonometric functions, and then evaluate the definite integral. We have
\[
\int_{0}^{1/2} \frac{dx}{\sqrt{1-x^2}} = \arcsin x \bigg|_{0}^{1/2} = \arcsin \frac{1}{2} - \arcsin 0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}. \nonumber
\]

Note that since the integrand is simply the derivative of \(\arcsin x\), we are really just using this fact to find the antiderivative here.

Exercise \(\PageIndex{1}\)

Find the indefinite integral using an inverse trigonometric function and substitution for \(\displaystyle \int \frac{dx}{\sqrt{9-x^2}}\).

Hint

Use the formula in the rule on integration formulas resulting in inverse trigonometric functions.
Answer

\[ \int \frac{dx}{\sqrt{9-x^2}} = \arcsin \left( \frac{x}{3} \right) + C \]

In many integrals that result in inverse trigonometric functions in the antiderivative, we may need to use substitution to see how to use the integration formulas provided above.

Example \((\PageIndex{2})\): Finding an Antiderivative Involving an Inverse Trigonometric Function using substitution

Evaluate the integral

\[ \int \frac{dx}{\sqrt{4-9x^2}}. \nonumber \]

Solution

Substitute \(( u=3x)\). Then \(( du=3\,dx)\) and we have

\[ \int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3} \int \frac{du}{\sqrt{4-u^2}}. \nonumber \]

Applying the formula with \(( a=2,)\) we obtain

\[ \int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3} \int \frac{du}{\sqrt{4-u^2}} = \frac{1}{3} \arcsin \left( \frac{u}{2} \right) + C = \arcsin \left( \frac{3x}{2} \right) + C. \nonumber \]

Exercise \((\PageIndex{2})\)

Find the antiderivative of \((\displaystyle \int \frac{dx}{\sqrt{1-16x^2}})\).

Hint

Substitute \(( u=4x)\).

Answer

\[ \int \frac{dx}{\sqrt{1-16x^2}} = \arcsin (4x) + C. \]

Example \((\PageIndex{3})\): Evaluating a Definite Integral

Evaluate the definite integral

\[ \int_{\sqrt{3}/2}^0 \frac{du}{\sqrt{1-u^2}}. \nonumber \]

Solution

The format of the problem matches the inverse sine formula. Thus,
Integrals Resulting in Other Inverse Trigonometric Functions

There are six inverse trigonometric functions. However, only three integration formulas are noted in the rule on integration formulas resulting in inverse trigonometric functions because the remaining three are negative versions of the ones we use. The only difference is whether the integrand is positive or negative. Rather than memorizing three more formulas, if the integrand is negative, simply factor out \(-1\) and evaluate the integral using one of the formulas already provided. To close this section, we examine one more formula: the integral resulting in the inverse tangent function.

Example \(\PageIndex{4}\): Finding an Antiderivative Involving the Inverse Tangent Function

Find the antiderivative of \(\displaystyle \int \frac{1}{9+x^2}\,dx\).

Solution

Apply the formula with \(a=3\). Then,

\[
\int \frac{dx}{9+x^2} = \frac{1}{3} \arctan \left(\frac{x}{3}\right) + C.
\]

Exercise \(\PageIndex{3}\)

Find the antiderivative of \(\displaystyle \int \frac{dx}{16+x^2}\).

Hint

Follow the steps in Example \(\PageIndex{4}\).

Answer

\(\displaystyle \int \frac{dx}{16+x^2} = \frac{1}{4} \arctan \left(\frac{x}{4}\right) + C\)

Example \(\PageIndex{5}\): Applying the Integration Formulas WITH SUBSTITUTION

Find an antiderivative of \(\displaystyle \int \frac{1}{1+4x^2}\,dx\).

Solution

Comparing this problem with the formulas stated in the rule on integration formulas resulting in inverse trigonometric functions, the integrand looks similar to the formula for \(\arctan u + C\). So we use substitution, letting \(u=2x\), then \(\,du=2\,dx\) and \(\int \frac{1}{1+u^2}\,du=\arctan u + C\). Then, we have

\[
\int \frac{1}{1+u^2}\,du = \arctan u + C.
\]

\[
\int \frac{1}{1+4x^2}\,dx = \frac{1}{2} \arctan (2x) + C.
\]
Exercise \(\PageIndex{4}\)

Use substitution to find the antiderivative of \(\displaystyle \int \frac{dx}{25+4x^2} \).

**Hint**

Use the solving strategy from Example \(\PageIndex{5}\) and the rule on integration formulas resulting in inverse trigonometric functions.

\[ \int \frac{dx}{25+4x^2} = \frac{1}{10} \arctan \left( \frac{2x}{5} \right) + C \]

**Example \(\PageIndex{6}\): Evaluating a Definite Integral**

Evaluate the definite integral \(\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2} \).

**Solution**  

Use the formula for the inverse tangent. We have

\[ \int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2} = \arctan x \bigg|_{\sqrt{3}/3}^{\sqrt{3}} = \arctan \left( \sqrt{3} \right) - \arctan \left( \frac{\sqrt{3}}{3} \right) = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6} \]

Exercise \(\PageIndex{5}\)

Evaluate the definite integral \(\int_{0}^{2} \frac{dx}{4+x^2} \).

**Hint**

Follow the procedures from Example \(\PageIndex{6}\) to solve the problem.

\[ \int_{0}^{2} \frac{dx}{4+x^2} = \frac{\pi}{8} \]

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**Simplifying the Integrand using Algebraic Methods**

It is common to be reluctant to manipulate the integrand of an integral; at first, our grasp of integration is tenuous and one may think that working with the integrand will improperly change the results. Integration by substitution works using a different logic: as long as equality is maintained, the integrand can be manipulated so that its form is easier to deal with. The next two examples demonstrate common ways in which using algebra first makes the integration easier to perform.

**Example \(\PageIndex{7}\): Integration by substitution: simplifying first using long division**

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Evaluate \( \int \frac{x^3+4x^2+8x+5}{x^2+2x+1} \, dx \).

**Solution**

One may try to start by setting \( \int (u) \) equal to either the numerator or denominator; in each instance, the result is not workable.

When dealing with rational functions (i.e., quotients made up of polynomial functions), it is an almost universal rule that everything works better when the degree of the numerator is less than the degree of the denominator. Hence we use polynomial long division whenever the integrand is a ratio of two polynomials and the degree of the numerator is not less than the degree of the denominator.

We skip the specifics of the steps, but note that when \( \int (x^2+2x+1) \) is divided into \( \int (x^3+4x^2+8x+5) \), it goes in \( \int (x+2) \) times with a remainder of \( \int (3x+3) \). Thus

\[
\frac{x^3+4x^2+8x+5}{x^2+2x+1} = x+2 + \frac{3x+3}{x^2+2x+1}.
\]

Integrating \( \int (x+2) \) is simple. The fraction can be integrated by setting \( \int (u = x^2+2x+1) \), giving \( \int (\frac{du}{2}) \). This is very similar to the numerator. Note that \( \int (\frac{du}{2}) \) and then consider the following:

\[
\int \frac{x^3+4x^2+8x+5}{x^2+2x+1} \, dx = \int (x+2) \, dx + \int \frac{3x+3}{x^2+2x+1} \, dx = \frac12x^2+2x+C_1 + \frac32\ln|x^2+2x+1| + C.
\]

In some ways, we "lucked out" in that after dividing, substitution was able to be done. In later sections we'll develop techniques for handling rational functions where substitution is not directly feasible.

Example \( \PageIndex{8} \): Integrals requiring multiple methods

Evaluate \( \int \frac{4-x}{\sqrt{16-x^2}} \, dx \).

**Solution**

This integral requires two different methods to evaluate it. We get to those methods by splitting up the integral:

\[
\int \frac{4-x}{\sqrt{16-x^2}} \, dx = \int \frac{4}{\sqrt{16-x^2}} \, dx - \int \frac{x}{\sqrt{16-x^2}} \, dx.
\]

The first integral is handled using a straightforward application of Theorem \( \PageIndex{2} \); the second integral is handled by substitution, with \( \int (u = 16-x^2) \). We handle each separately.

\[
\int \frac{4}{\sqrt{16-x^2}} \, dx = 4\arcsin\frac{x}{4} + C.
\]

\[
\int \frac{x}{\sqrt{16-x^2}} \, dx: \text{Set } u = 16-x^2, \text{ so } \int (\frac{du}{2}) \text{ and } (\frac{dx}{-du/2}). \text{ We have}
\]

\[
\int \frac{dx}{\sqrt{16-x^2}} \, dx = \int \frac{dx}{\sqrt{16-x^2}} \, dx + \frac{1}{\sqrt{16-x^2}} + C.
\]
Combining these together, we have
\[
\int \frac{4-x}{\sqrt{16-x^2}}\ dx = 4\arcsin\frac x4 + \sqrt{16-x^2}+C.
\]

Using Completing the Square in Integration

Sometimes, we will see polynomials in the denominator that are quadratic in form and which we can use the process of completing the square to rewrite them in a form that we will recognize as the derivative of an inverse trigonometric function.

Example \(\PageIndex{9}\): Integrating by substitution: completing the square

Evaluate \(\displaystyle \int \frac{1}{x^2-4x+13}\ dx\).

**Solution**

Initially, this integral seems to have nothing in common with the integrals in Theorem \(\PageIndex{2}\). As it lacks a square root, it almost certainly is not related to arcsine or arccscant. It is, however, related to the arctangent function.

We see this by **completing the square** in the denominator. We give a brief reminder of the process here.

Start with a quadratic with a leading coefficient of 1. It will have the form of \((x^2 + bx + c)\). Take 1/2 of \(b\), square it, and add/subtract it back into the expression. I.e.,

\[
\begin{align}
 x^2+bx+ c &= \underbrace{x^2 + bx + \left(\frac{b^2}{4}\right)}_{(x+b/2)^2} - \frac{b^2}{4} + c \\
 &= (x+b/2)^2 + c - \frac{b^2}{4}
\end{align}
\]

In our example, we take half of \((-4)\) and square it, getting \(4\). We add/subtract it into the denominator as follows:

\[
\begin{align}
 \frac{1}{x^2-4x+13} &= \frac{1}{\underbrace{x^2-4x+4}_{(x-2)^2}-4+13} \\
 &= \frac{1}{(x-2)^2 + 9}
\end{align}
\]

We can now integrate this using the arctangent rule. Technically, we need to substitute first with \(u=x-2\), but we can employ Key Idea 10 instead. Thus we have

\[
\int \frac{1}{x^2-4x+13}\ dx = \int \frac{1}{(x-2)^2+9}\ dx = \frac{1}{3}\arctan\frac{x-2}{3}+C.
\]

Key Concepts

- Formulas for derivatives of inverse trigonometric functions developed in Derivatives of Exponential and Logarithmic Functions lead directly to integration formulas involving inverse trigonometric functions.
- Use the formulas listed in the rule on integration formulas resulting in inverse trigonometric functions to match up the correct format and make alterations as necessary to solve the problem.
- Substitution is often required to put the integrand in the correct form.
Key Equations

- **Integrals That Produce Inverse Trigonometric Functions**
  
  \[ \int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \left( \frac{u}{a} \right) + C \]
  
  \[ \int \frac{du}{a^2+u^2} = \frac{1}{a} \arctan \left( \frac{u}{a} \right) + C \]
  
  \[ \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} \text{arcsec} \left( \frac{|u|}{a} \right) + C \]

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- Paul Seeburger (Monroe Community College) made some added textual clarifications and edits on the OpenStax content in this section, and he also added and edited the last two subsections from Apex Calculus, Section 6.1.
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