9.3: The Divergence and Integral Tests

Learning Objectives

- Use the divergence test to determine whether a series converges or diverges.
- Use the integral test to determine the convergence of a series.
- Estimate the value of a series by finding bounds on its remainder term.

In the previous section, we determined the convergence or divergence of several series by explicitly calculating the limit of the sequence of partial sums $\{S_k\}$. In practice, explicitly calculating this limit can be difficult or impossible. Luckily, several tests exist that allow us to determine convergence or divergence for many types of series. In this section, we discuss two of these tests: the divergence test and the integral test. We will examine several other tests in the rest of this chapter and then summarize how and when to use them.

Divergence Test

For a series $\sum_{n=1}^{\infty}a_n$ to converge, the $n^{th}$ term $a_n$ must satisfy $a_n \to 0$ as $n \to \infty$. Therefore, from the algebraic limit properties of sequences,

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} (S_k - S_{k-1}) = \lim_{k \to \infty} S_k - \lim_{k \to \infty} S_{k-1} = S - S = 0.$$

Therefore, if $\sum_{n=1}^{\infty}a_n$ converges, the $n^{th}$ term $a_n \to 0$ as $n \to \infty$. An important consequence of this fact is the following statement:

If $a_n \not\to 0$ as $n \to \infty$, $\sum_{n=1}^{\infty}a_n$ diverges.
This test is known as the divergence test because it provides a way of proving that a series diverges.

**Definition: The Divergence Test**

If \( \lim_{n \to \infty} a_n = c \neq 0 \) or \( \lim_{n \to \infty} a_n \) does not exist, then the series \( \sum_{n=1}^{\infty} a_n \) diverges.

It is important to note that the converse of this theorem is not true. That is, if \( \lim_{n \to \infty} a_n = 0 \), we cannot make any conclusion about the convergence of \( \sum_{n=1}^{\infty} a_n \).

For example, \( \lim_{n \to 0} \frac{1}{n} = 0 \), but the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges. In this section and the remaining sections of this chapter, we show many more examples of such series.

Consequently, although we can use the divergence test to show that a series diverges, we cannot use it to prove that a series converges. Specifically, if \( a_n \to 0 \), the divergence test is inconclusive.

**Example \( \PageIndex{1} \): Using the divergence test**

For each of the following series, apply the divergence test. If the divergence test proves that the series diverges, state so. Otherwise, indicate that the divergence test is inconclusive.

a. \( \sum_{n=1}^{\infty} \frac{n}{3n-1} \)

b. \( \sum_{n=1}^{\infty} \frac{1}{n^3} \)

c. \( \sum_{n=1}^{\infty} e^{1/n^2} \)

**Solution**

a. Since \( \lim_{n \to \infty} \frac{n}{3n-1} = \frac{1}{3} \neq 0 \), by the divergence test, we can conclude that \( \sum_{n=1}^{\infty} \frac{n}{3n-1} \) diverges.

b. Since \( \lim_{n \to \infty} \frac{1}{n^3} = 0 \), the divergence test is inconclusive.

c. Since \( \lim_{n \to \infty} e^{1/n^2} = 1 \neq 0 \), by the divergence test, the series \( \sum_{n=1}^{\infty} e^{1/n^2} \) diverges.

**Exercise \( \PageIndex{1} \)**

What does the divergence test tell us about the series \( \sum_{n=1}^{\infty} \cos(1/n^2) \)?

**Hint**

Look at \( \lim_{n \to \infty} \cos(1/n^2) \).

**Answer**

The series diverges.
Integral Test

In the previous section, we proved that the harmonic series diverges by looking at the sequence of partial sums \( \{ S_k \} \) and showing that \( S_{2^k} > 1 + k/2 \) for all positive integers \( k \). In this section we use a different technique to prove the divergence of the harmonic series. This technique is important because it is used to prove the divergence or convergence of many other series. This test, called the integral test, compares an infinite sum to an improper integral. It is important to note that this test can only be applied when we are considering a series whose terms are all positive.

Figure \( \PageIndex{1} \): The sum of the areas of the rectangles is greater than the area between the curve \( f(x) = 1/x \) and the \( y \)-axis for \( x \geq 1 \). Since the area bounded by the curve is infinite (as calculated by an improper integral), the sum of the areas of the rectangles is also infinite.

To illustrate how the integral test works, use the harmonic series as an example. In Figure \( \PageIndex{1} \), we depict the harmonic series by sketching a sequence of rectangles with areas \( 1, 1/2, 1/3, 1/4, \ldots \) along with the function \( f(x) = 1/x \).

From the graph, we see that

\[
\sum_{n=1}^k \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} > \int_{1}^{k+1} \frac{1}{x} \, dx.
\]

Therefore, for each \( k \), the \( k \)th partial sum \( \{ S_k \} \) satisfies

\[
\begin{align*}
S_k = \sum_{n=1}^k \frac{1}{n} > \int_{1}^{k+1} \frac{1}{x} \, dx &= \ln x \bigg|_{1}^{k+1} \\
&= \ln (k+1) - \ln 1 \\
&= \ln (k+1).
\end{align*}
\]

Since \( \lim_{k \to \infty} \ln (k+1) = \infty \), we see that the sequence of partial sums \( \{ S_k \} \) is unbounded. Therefore, \( \{ S_k \} \) diverges, and, consequently, the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) also diverges.
Figure \(\PageIndex{2}\): The sum of the areas of the rectangles is less than the sum of the area of the first rectangle and the area between the curve \( f(x)=1/x^2 \) and the \( x\)-axis for \( x\geq1 \). Since the area bounded by the curve is finite, the sum of the areas of the rectangles is also finite.

Now consider the series \( \displaystyle \sum_{n=1}^\infty \frac{1}{n^2} \). We show how an integral can be used to prove that this series converges. In Figure \( \PageIndex{2} \), we sketch a sequence of rectangles with areas \( 1, 1/2^2, 1/3^2, \ldots \) along with the function \( f(x)=1/x^2 \). From the graph we see that

\[
\sum_{n=1}^k \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{k^2} < 1 + \int_1^k \frac{1}{x^2} \, dx.
\]

Therefore, for each \( k \), the \( k\)th partial sum \( S_k \) satisfies

\[
\begin{align*}
S_k &= \sum_{n=1}^k \frac{1}{n^2} < 1 + \int_1^k \frac{1}{x^2} \, dx \\
&= 1 - \left. \frac{1}{x} \right|_1^k \\
&= 2 - \frac{1}{k} < 2.
\end{align*}
\]

We conclude that the sequence of partial sums \( \{ S_k \} \) is bounded. We also see that \( \{ S_k \} \) is an increasing sequence:

\[S_k = S_{k-1} + \frac{1}{k^2}\]

for \( k \geq 2 \).

Since \( \{ S_k \} \) is increasing and bounded, by the **Monotone Convergence Theorem**, it converges. Therefore, the series \( \displaystyle \sum_{n=1}^\infty \frac{1}{n^2} \) converges.
Figure \(\PageIndex{3}\): (a) If we can inscribe rectangles inside a region bounded by a curve \( y=f(x) \) and the \( x \)-axis, and the area bounded by those curves for \( x\geq 1 \) is finite, then the sum of the areas of the rectangles is also finite. (b) If a set of rectangles circumscribes the region bounded by \( y=f(x) \) and the \( x \)-axis for \( x\geq 1 \) and the region has infinite area, then the sum of the areas of the rectangles is also infinite.

We can extend this idea to prove convergence or divergence for many different series. Suppose \( \displaystyle \sum_{n=1}^{\infty} a_n \) is a series with positive terms \( a_n \) such that there exists a continuous, positive, decreasing function \( f \) where \( f(n)=a_n \) for all positive integers. Then, as in Figure \(\PageIndex{3a}\), for any integer \( k \), the \( k^{\text{th}} \) partial sum \( S_k \) satisfies

\[
[S_k=a_1+a_2+a_3+\ldots+a_k<\int_1^k f(x)\,dx<1+\int_1^{\infty} f(x)\,dx].
\]

Therefore, if \( \displaystyle \int_1^{\infty} f(x)\,dx \) converges, then the sequence of partial sums \( \{ S_k \} \) is bounded. Since \( \{ S_k \} \) is an increasing sequence, if it is also a bounded sequence, then by the Monotone Convergence Theorem, it converges. We conclude that if \( \displaystyle \int_1^{\infty} f(x)\,dx \) converges, then the series \( \displaystyle \sum_{n=1}^{\infty} a_n \) also converges. On the other hand, from Figure \(\PageIndex{3b}\), for any integer \( k \), the \( k^{\text{th}} \) partial sum \( S_k \) satisfies

\[
[S_k=a_1+a_2+a_3+\ldots+a_k>\int_{k+1}^{\infty} f(x)\,dx].
\]

If

\[
[ \lim_{k\to\infty} \int_{k+1}^{\infty} f(x)\,dx=\infty,]
\]

then \( \{ S_k \} \) is an unbounded sequence and therefore diverges. As a result, the series \( \displaystyle \sum_{n=1}^{\infty} a_n \) also diverges. Since \( f \) is a positive function, if \( \displaystyle \int_1^{\infty} f(x)\,dx \) diverges, then

\[
[ \lim_{k\to\infty} \int_{k+1}^{\infty} f(x)\,dx=\infty,]
\]

We conclude that if \( \displaystyle \int_1^{\infty} f(x)\,dx \) diverges, then \( \displaystyle \sum_{n=1}^{\infty} a_n \) diverges.

Definition: The Integral Test

Suppose \( \displaystyle \sum_{n=1}^{\infty} a_n \) is a series with positive terms \( a_n \). Suppose there exists a function \( f \) and a positive integer \( N \) such that the following three conditions are satisfied:

i. \( f \) is continuous,

ii. \( f \) is decreasing, and

iii. \( f(n)=a_n \) for all integers \( n\geq N \).

Then

\[
[\sum_{n=1}^{\infty} a_n \ \text{converges}]
\]

and
\[
\int_0^\infty Nf(x)\,dx \nonumber
\]

both converge or both diverge (Figure \(\PageIndex{3}\)).

Although convergence of \(\int_0^\infty Nf(x)\,dx\) implies convergence of the related series \(\sum_{n=1}^\infty a_n\), it does not imply that the value of the integral and the series are the same. They may be different, and often are. For example,

\[
\sum_{n=1}^\infty \left(\frac{1}{e}\right)^n = \frac{1}{e} + \left(\frac{1}{e}\right)^2 + \left(\frac{1}{e}\right)^3 + \ldots
\]

is a geometric series with initial term \(a=1/e\) and ratio \(r=1/e\), which converges to

\[
\frac{1/e}{1 - (1/e)} = \frac{1}{e - 1}.
\]

However, the related integral \(\int_1^\infty \left(\frac{1}{e}\right)^x\,dx\) satisfies

\[
\int_1^\infty \left(\frac{1}{e}\right)^x\,dx = \lim_{b \to \infty} \int_1^b \left(\frac{1}{e}\right)^x\,dx = \lim_{b \to \infty} \left[ -\frac{1}{e^x} \right]_1^b = \lim_{b \to \infty} \left[ -\frac{1}{e^b} + \frac{1}{e^1} \right] = \frac{1}{e}.
\]

Example \(\PageIndex{2}\): Using the Integral Test

For each of the following series, use the integral test to determine whether the series converges or diverges.

a. \(\sum_{n=1}^\infty \frac{1}{n^3}\)

b. \(\sum_{n=1}^\infty \frac{1}{\sqrt{2n-1}}\)

Solution

a. Compare \(\sum_{n=1}^\infty \frac{1}{n^3}\) and \(\int_1^\infty \frac{1}{x^3}\,dx\).

We have

\[
\int_1^\infty \frac{1}{x^3}\,dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^3}\,dx = \lim_{b \to \infty} \left[ -\frac{1}{2x^2} \right]_1^b = \lim_{b \to \infty} \left[ -\frac{1}{2b^2} + \frac{1}{2} \right] = \frac{1}{2}.
\]

Thus the integral \(\int_1^\infty \frac{1}{x^3}\,dx\) converges, and therefore so does the series

\(\sum_{n=1}^\infty \frac{1}{n^3}\).

b. Compare

\(\sum_{n=1}^\infty \frac{1}{\sqrt{2n-1}}\) and \(\int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx\).
Since \[
\int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx = \lim_{b \to \infty} \int_1^b \frac{1}{\sqrt{2x-1}}\,dx = \lim_{b \to \infty} \left[ \frac{1}{\sqrt{2x-1}} \right]_1^b = \lim_{b \to \infty} \left( \sqrt{2b-1} - 1 \right) = \infty,
\]
the integral \[
\int_1^\infty \frac{1}{\sqrt{2x-1}}\,dx
\]
diverges, and therefore
\[
\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}
\]
diverges.

Exercise \( \PageIndex{2} \)

Use the integral test to determine whether the series \[
\sum_{n=1}^{\infty} \frac{n}{3n^2+1}
\]
converges or diverges.

**Hint**

Compare to the integral \[
\int_1^\infty \frac{x}{3x^2+1}\,dx.
\]

**Answer**

The series diverges.

**The \(p\)-Series**

The harmonic series \[
\sum_{n=1}^{\infty} \frac{1}{n}
\]
and the series \[
\sum_{n=1}^{\infty} \frac{1}{n^2}
\]
are both examples of a type of series called a p-series.

**Definition:** \(p\)-series

For any real number \(p\), the series
\[
\sum_{n=1}^{\infty} \frac{1}{n^p}
\]
is called a \(p\)-series.

We know the \(p\)-series converges if \(p=2\) and diverges if \(p=1\). What about other values of \(p\)? In general, it is difficult, if not impossible, to compute the exact value of most \(p\)-series. However, we can use the tests presented thus far to prove whether a \(p\)-series converges or diverges.

If \(p=0\), then \(1/n^p\to\infty\) and if \(p=1\), then \(1/n^p\to1\). Therefore, by the divergence test,
\[
\sum_{n=1}^{\infty} \frac{1}{n^p}
\]
diverges if \(p\leq0\).
If \((p > 0)\), then \(f(x) = 1/x^p\) is a positive, continuous, decreasing function. Therefore, for \((p > 0)\), we use the integral test, comparing

\[\sum_{n=1}^{\infty} \frac{1}{n^p}\] and \[\int_1^{\infty} \frac{1}{x^p} \, dx.\]

We have already considered the case when \((p = 1)\). Here we consider the case when \((p > 0; p \neq 1)\). For this case,

\[\int_1^{\infty} \frac{1}{x^p} \, dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{x^p} \, dx = \lim_{b \to \infty} \frac{1}{1-p} x^{1-p} \bigg|_1^b = \lim_{b \to \infty} \frac{1}{1-p} [b^{1-p} - 1].\]

Because

\(b^{1-p} \to 0\) if \((p > 1)\) and \(b^{1-p} \to \infty\) if \((p < 1)\),

we conclude that

\[\int_1^{\infty} \frac{1}{x^p} \, dx = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1; \\ \infty, & \text{if } p < 1. \end{cases}\]

Therefore, \(\sum_{n=1}^{\infty} 1/n^p\) converges if \((p > 1)\) and diverges if \((0 < p < 1)\).

In summary,

\[\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges} & \text{if } p > 1; \\ \text{diverges} & \text{if } p \leq 1. \end{cases}\]

Example \(\PageIndex{3}\): Testing for Convergence of p-series

For each of the following series, determine whether it converges or diverges.

a. \(\sum_{n=1}^{\infty} \frac{1}{n^4}\)

b. \(\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}\)

Solution

a. This is a \((p)\)-series with \((p = 4 > 1)\), so the series converges.

b. Since \((p = 2/3 < 1)\), the series diverges.

Exercise \(\PageIndex{3}\)

Does the series \(\sum_{n=1}^{\infty} \frac{1}{n^4}\) converge or diverge?

Hint

\((p = 5/4)\)

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**Answer**

The series converges.

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**Estimating the Value of a Series**

Suppose we know that a series \( \sum_{n=1}^{\infty}a_n \) converges and we want to estimate the sum of that series. Certainly we can approximate that sum using any finite sum \( \sum_{n=1}^{N}a_n \) where \( N \) is any positive integer. The question we address here is, for a convergent series \( \sum_{n=1}^{\infty}a_n \), how good is the approximation \( \sum_{n=1}^{N}a_n \)?

More specifically, if we let

\[
R_N = \sum_{n=1}^{\infty}a_n - \sum_{n=1}^{N}a_n
\]

be the remainder when the sum of an infinite series is approximated by the \( N \)th partial sum, how large is \( R_N \)? For some types of series, we are able to use the ideas from the integral test to estimate \( R_N \).

**Note \( \PageIndex{1} \): Remainder Estimate from the Integral Test**

Suppose \( \sum_{n=1}^{\infty}a_n \) is a convergent series with positive terms. Suppose there exists a function \( f \) satisfying the following three conditions:

i. \( f \) is continuous,

ii. \( f \) is decreasing, and

iii. \( f(n) = a_n \) for all integers \( n \geq 1 \).

Let \( S_N \) be the \( (N) \) partial sum of \( \sum_{n=1}^{\infty}a_n \). For all positive integers \( N \),

\[
[S_N + \int_{N+1}^{\infty}f(x)\,dx] < \sum_{n=1}^{\infty}a_n < [S_N + \int_{N}^{\infty}f(x)\,dx].
\]

In other words, the remainder \( R_N = \sum_{n=1}^{\infty}a_n - S_N \) satisfies the following estimate:

\[
[S_N + \int_{N+1}^{\infty}f(x)\,dx] < R_N < [S_N + \int_{N}^{\infty}f(x)\,dx].
\]

This is known as the remainder estimate.

We illustrate Note \( \PageIndex{1} \) in Figure \( \PageIndex{4} \). In particular, by representing the remainder \( R_N \) as the sum of areas of rectangles, we see that the area of those rectangles is bounded above by \( \int_{N+1}^{\infty}f(x)\,dx \) and bounded below by \( \int_{N}^{\infty}f(x)\,dx \). In other words,

\[
[R_N = a_{N+1} + a_{N+2} + a_{N+3} + \ldots] > [\int_{N+1}^{\infty}f(x)\,dx] \]

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\[ R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_0^\infty f(x) \, dx. \]

We conclude that
\[ \int_a^\infty f(x) \, dx < R_N < \int_a^\infty f(x) \, dx. \]

Since
\[ \sum_{n=1}^\infty a_n = S_N + R_N, \]
where \( S_N \) is the \((N^\text{th})\) partial sum, we conclude that
\[ S_N + \int_a^\infty f(x) \, dx < \sum_{n=1}^\infty a_n < S_N + \int_a^\infty f(x) \, dx. \]

Figure (\( \PageIndex{4} \)): Given a continuous, positive, decreasing function \( f \) and a sequence of positive terms \( a_n \) such that \( a_n = f(n) \) for all positive integers \( n \), (a) the areas \( \sum_{n=1}^\infty a_n = S_N + R_N \), or (b) the areas \( \sum_{n=1}^\infty a_n = S_N + R_N \). Therefore, the integral is either an overestimate or an underestimate of the error.

Example (\( \PageIndex{4} \)): Estimating the Value of a Series

Consider the series \( \sum_{n=1}^\infty \frac{1}{n^3} \).

a. Calculate \( \sum_{n=1}^{10} \frac{1}{n^3} \) and estimate the error.

b. Determine the least value of \( N \) necessary such that \( \sum_{n=1}^N \frac{1}{n^3} \) will estimate \( \sum_{n=1}^\infty \frac{1}{n^3} \) to within \( 0.001 \).

Solution

a. Using a calculating utility, we have
\[ \sum_{n=1}^{10} \frac{1}{n^3} \approx 1.19753. \]

By the remainder estimate, we know
\[ R_N < \int_N^\infty \frac{1}{x^3} \, dx. \]
We have
\[
\int_{10}^{\infty} \frac{1}{x^3} \, dx = \lim_{b \to \infty} \int_{10}^{b} \frac{1}{x^3} \, dx = \lim_{b \to \infty} \left[ -\frac{1}{2x^2} \right]_{10}^{b} = \lim_{b \to \infty} \left( -\frac{1}{2b^2} + \frac{1}{200} \right) = \frac{1}{200}.
\]

Therefore, the error is \(| R_{10} | < 1/2(10)^2 = 0.005 |\).

b. Find \( N \) such that \(| R_N | < 0.001 \). In part a. we showed that \(| R_N | < 1/2N^2 \). Therefore, the remainder \(| R_N | < 0.001 \) as long as \(| 1/2N^2 | < 0.001 \). That is, we need \(| 2N^2 | > 1000 \). Solving this inequality for \(| N |\), we see that we need \(| N | > 22.36 \). To ensure that the remainder is within the desired amount, we need to round up to the nearest integer. Therefore, the minimum necessary value is \(| N = 23 |\).

Exercise \(| \PageIndex{4} \)|

For \( \sum_{n=1}^{\infty} \frac{1}{n^4} \), calculate \( S_5 \) and estimate the error \( R_5 \).

Hint

Use the remainder estimate \( R_N < \int_{N}^{\infty} \frac{1}{x^4} \, dx \).

Answer

\(| S_5 | \approx 1.09035, R_5 | < 0.00267 |\)

Key Concepts

- If \( \lim_{n\to\infty} a_n \neq 0 \) then the series \( \sum_{n=1}^{\infty} a_n \) diverges.
- If \( \lim_{n\to\infty} a_n = 0 \) the series \( \sum_{n=1}^{\infty} a_n \) may converge or diverge.
- If \( \sum_{n=1}^{\infty} a_n \) is a series with positive terms \( a_n \) and \( f \) is a continuous, decreasing function such that \( f(n) = a_n \) for all positive integers \( n \), then

\[ \int_{\sum_{n=1}^{\infty} a_n}^{\infty} f(x) \, dx \]

either both converge or both diverge. Furthermore, if \( \sum_{n=1}^{\infty} a_n \) converges, then the \( N^{th} \) partial sum approximation \( S_N \) is accurate up to an error \( R_N \) where \( R_N \) is given by \( \int_{N}^{\infty} f(x) \, dx \).

- The \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges if \( p > 1 \) and diverges if \( p \leq 1 \).

Key Equations

- Divergence test
If \( a_n \neq 0 \) as \( n \to \infty \), \( \sum_{n=1}^{\infty} a_n \) diverges.

- **p-series**

\[
\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} 
\text{converges}, & \text{if } p > 1 \\
\text{diverges}, & \text{if } p \leq 1
\end{cases}
\]

- **Remainder estimate from the integral test**

\[
\int_{N+1}^{\infty} f(x) \, dx < R_N < \int_{N}^{\infty} f(x) \, dx
\]

---

**Glossary**

- **Divergence test**
  
  If \( \lim_{n \to \infty} a_n \neq 0 \), then the series \( \sum_{n=1}^{\infty} a_n \) diverges.

- **Integral test**
  
  For a series \( \sum_{n=1}^{\infty} a_n \) with positive terms \( a_n \), if there exists a continuous, decreasing function \( f \) such that \( f(n) = a_n \) for all positive integers \( n \), then
  
  \[
  \left[ \sum_{n=1}^{\infty} a_n \right] \text{ and } \left[ \int_{1}^{\infty} f(x) \, dx \right]
  \]
  
  either both converge or both diverge

- **p-series**
  
  A series of the form \( \sum_{n=1}^{\infty} \frac{1}{n^p} \)

- **Remainder estimate**
  
  For a series \( \sum_{n=1}^{\infty} a_n \) with positive terms \( a_n \) and a continuous, decreasing function \( f \) such that \( f(n) = a_n \) for all positive integers \( n \), the remainder \( R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \) satisfies the following estimate:
  
  \[
  \left[ \int_{N+1}^{\infty} f(x) \, dx \right] < R_N < \left[ \int_{N}^{\infty} f(x) \, dx \right]
  \]

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