Hall-Littlewood Polynomials

We know the schur basis, and many more, for the ring of symmetric functions over a field \( \mathbb{F} \). The next step of generalization is consider the field \( \mathbb{F}(t) \), and twist a little bit the inner product. In contrast with Macdonald polynomials, we can give a closed expression for Hall-Littlewood polynomials

**Definition and first properties**

First we need the following \( t \)-analogues

\[
[k]_t := \frac{1-t^k}{1-t} = 1 + t + t^2 + \cdots + t^{k-1}
\]

\[
[k]_t! := [k]_t[k-1]_t\cdots[1]_t
\]

Then the Hall-Littlewood polynomial \( P_{\lambda}(x;t) \) in \( n \) variables is given by the following formula

\[
P_{\lambda}(x;t) = \frac{1}{\prod_{i\geq0} [\alpha_i]_t!} \sum_{w \in S_n} w \left( x^{\lambda} \frac{\prod_{i<j}(1-tx_j/x_i)}{\prod_{i<j}(1-x_j/x_i)} \right)
\]

Where \( \lambda = (1^{\alpha_1}, 2^{\alpha_2}, \cdots) \) and \( \alpha_0 \) is such that \( \sum_{i \geq 0} \alpha_i = n \)

Note that when \( t=0 \) the denominator \( \prod_{i \geq 0} [\alpha_i]_t! \) goes away and we get precisely the Weyl's character formula for the schur functions, so

\[
P_{\lambda}(x,0) = s_\lambda(x)
\]
at \( t=1 \) the products inside cancel and we get the usual monomial functions

\[
P_{\lambda}(x,1) = m_\lambda(x)
\]

The Hall-Littlewood polynomials will form a basis, then we can expand schur in this new basis. The "Kostka-Foulkes polynomials" \( K_{\lambda\mu}(t) \) are defined by

\[
s_\lambda(x) = \sum_{\mu} K_{\lambda\mu}(t) P_{\lambda}(x; t)
\]

They don't deserve the name polynomials yet, because so far we just know that they are rational functions in \( t \). But we will see why they're actual polynomials.

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**Definition with raising operators**

Define the *Jing Operators* as \( t \) deformations of the Bernstein operator in the following way

\[
S^t_m f = [u^m]f[X+(t-1)u^{-1}][\Omega[uX]]
\]

and their modified version

\[
\tilde{S}^t_m f = [u^m]f[X-u^{-1}][\Omega[(1-t)uX]]
\]

which are related by

\[
\tilde{S}^t_m = \Pi_{(1-t)}S^t_m\Pi^{-1}_{(1-t)}
\]

where \( \Pi_{(1-t)} \) is the operator with the plethystic substituition \( f \to f[X(1-t)] \), and \( \Pi^{-1}_{(1-t)} \) is its inverse, namely \( f \to f[X/(1-t)] \).

Analogously to the schur functions now defined the *transformed Hall-Littlewood polynomials* as

\[
H_\mu(x; t) = S^t_{\mu_1}S^t_{\mu_2}\cdots S^t_{\mu_l}(1)
\]

And if we set \( Q_\mu(x; t) = H_\mu((1-t)X; t) \) we get

\[
Q_\mu(z; t) = \tilde{S}^t_{\mu_1}\tilde{S}^t_{\mu_2}\cdots \tilde{S}^t_{\mu_l}(1)
\]

Recall that the Bernstein operators added one part to a partition. This new operators behave in a more complicated way, but of similar spirit

**Theorem: Jing Operators**

If \( m \geq \mu_1 \gamma \) and \( \lambda \geq \mu \) then

\[
S^t_{\lambda} \in \mathbb{Z}[t] \{ s_\gamma : \gamma \geq (m, \mu) \}
\]
Moreover, \( s_{(m, \lambda)} \) appears with coefficient 1.

The last part is saying something similar to the previous situation, we will get the Schur function with an additional part \( m \) added, but the theorem is saying that we get also polynomial combinations of other Schur functions.

By repeated use of the theorem we can conclude that

\[
H_{\mu}(x; t) = \sum_{\lambda \geq \mu} C_{\lambda \mu}(t) s_{\lambda}(x)
\]

where \( C_{\lambda \mu}(t) \) are polynomials with \( C_{\mu \mu}(t) = 1 \).

That means that we have upper unitriangularity with respect to the Schur basis.

We have analogous statements for \( Q \) (although with different proof!)

**Theorem: Modified Jing Operators**

If \( m \geq \mu_1 \gamma \) then

\[
\tilde{S}^t_m s_\lambda \in \mathbb{Z}[t] \{ s_\gamma : \gamma \leq (m, \lambda) \}
\]

Moreover, \( s_{(m, \lambda)} \) appears with coefficient \( 1 - t^{\alpha} \) where \( \alpha \) is the multiplicity of \( m \) as a part of \( (m, \lambda) \).

Again by repeated use of the theorem we can conclude that

\[
Q_{\mu}(x; t) = \sum_{\lambda \leq \mu} B_{\lambda \mu}(t) s_{\lambda}(x)
\]

where \( B_{\lambda \mu}(t) \) are polynomials with \( B_{\mu \mu}(t) = (1-t)^{l(\mu)} \prod_{i \geq 0} \alpha_i^! \).

Which means that we have lower triangularity (but with a messier diagonal elements) with respect to the Schur basis.

The operator \( \Pi \) is self adjoint for the inner product, i.e. we have

\[
\langle f, g[(1-t)X] \rangle = \langle f[(1-t)X], g \rangle
\]

By the opposite triangularities of \( H \) and \( Q = H[(1-t)X] \) we have that if \( \langle f, g[(1-t)X] \rangle \neq 0 \) then \( \langle g \mu \rangle = \mu \langle f(1-t)X \rangle \). Passing the \( (1-t) \) to the other side, we obtain the opposite conclusion \( \langle f \mu \rangle = \mu \langle g[(1-t)X] \rangle \) and hence \( \langle f \mu \rangle = \mu \langle g[(1-t)X] \rangle \). Which implies the following claim

The transformed Hall-Littlewood polynomials are orthogonal with respect to the inner product \( \langle f, g[(1-t)X] \rangle \) and their self inner products are given by
Now everything fits smoothly

Really. First, from the definition of $\langle Q \rangle$ one can get the following formula by induction

\[
\langle Q_{\lambda}(x;t) = (1-t)^{l(\lambda)}\prod_{i \geq 0} [\alpha_i]_t \sum_{w \in S_n} w(x^{\lambda}\sum_{j<i}(1-tx_i/j-x_j)) \rangle = Q_{\lambda}(x;t)
\]

The relation with the original Hall-Littlewood polynomials is

\[
\langle P_{\lambda}(x;t) = \frac{Q_{\lambda}(x;t)}{(1-t)^{l(\lambda)}\prod_{i \geq 0} [\alpha_i]_t!} \rangle
\]

Note that the denominator is precisely the self inner product of the $\langle H \rangle$ in the inner product $\langle f,g[(1-t)X]\rangle$. Classically something a bit different is defined

\[
\langle f,g \rangle_t = \langle f,g[X/(1-t)] \rangle
\]

In this product, the basis $\langle P_{\lambda}(\lambda) \rangle$ and $\langle Q_{\lambda}(\lambda) \rangle$ are orthogonal and furthermore, they are dual! So recall that we defined the Kostka - Foulkes polynomial as

\[
\langle s_{\lambda}(x) = \sum_{\mu} K_{\lambda\mu}(t) P_{\lambda}(x) \rangle
\]

By taking inner products, and using the duality just mentioned we arrive at

\[
\langle K_{\lambda\mu}(t) = \langle s_{\lambda}, Q_{\mu} \rangle = \langle s_{\lambda}, H_{\mu} \rangle = C_{\lambda\mu}(t) \rangle
\]

But that last coefficient is equal to our previously defined polynomials $\langle C_{\lambda\mu}(t) \rangle$, showing that the Kostka-Foulkes polynomials are in fact polynomials.

Positivity of Kostka-Foulkes polynomials

It turns out that they are not just integer polynomials, but their coefficients are positive. It may not sound very interesting to show that a quantity is positive, but usually the question is implicitly asking for an interpretation. There are many different approaches here, all far from trivial. Let's briefly review them.

Representation theory

The work of Hotta, Lusztig, and Springer showed deep connections with representation theory. I cannot say more than a few words: They relate the Kostka-Foulkes polynomials, and a variation of them, called "cocharge" Kostka-Foulkes polynomials to some hardcore math where the keywords are "Unipotent Characters, local intersection homology, Springer fiber and perverse sheaves".
The important point is that they found a ring, the cohomology ring of the Springer fiber, whose Frobenius series is given by the cocharge transformed Hall-Littlewood polynomials, implying they expand schur positively.

### Combinatorics of Tableaux

Lascoux and Schutzenberger proved the following simple and elegant formula, that gives a concrete meaning to each coefficient

\[
K_{\lambda \mu}(t) = \sum_T t^{c(T)}
\]

the sum is over all SSYT of shape \( \lambda \) and content \( \mu \). The new definition is the "charge" \( c(T) \) which is easier to define in terms of cocharge \( cc(T) \) which is an invariant characterized by

1. Cocharge is invariant under jeu-de-taquin slides
2. Suppose the shape of \( T \) is disconnected, say \( T = X \cup Y \) with \( X \) above and left of \( Y \), and no entry of \( X \) is equal to 1. Then \( S = Y \cup X \), obtained by swapping, has \( cc(S) = |X| - cc(X) \)
3. If \( T \) is a single row, then \( cc(T) = 0 \)

And then \( c(T) = n(\mu) - cc(T) \). The existence of such an invariant requires proof. There is a process to compute the cocharge called "catabolism".

### Alternative description using tableaux

Kirillov and Reshetikhin gave the following formula

\[
K_{\lambda \mu}(t) = \sum_{\upsilon} (\lambda, \mu, \upsilon)
\]

where the sum is over all \((\lambda, \mu, \upsilon)\) - admissible configurations \((\upsilon)\).

Complicated as it seems, this expression has clearly positive coefficients. The origin of this formula is from a technique in mathematical physics known as "Bethe ansatz", which is used to produced highest weight vectors for some tensor products. The theorem is relating \( K_{\lambda \mu}(t) \) with the enumeration of highest weight vectors in \( V_{\mu_1} \otimes \cdots \otimes V_{\mu_r} \) by a quantum number. For more info, stay tuned, probably Anne has something to say about in class.

### Commutative Algebra

This may be the less technical. Garsia and Procesi simplified the first proof by giving a down to earth interpretation of the cohomology ring of the springer fiber \((R_{\mu})\). Now the action happens inside the polynomial ring \((C[x] = C[x_1, x_2, \cdots, x_n])\). And
\[ R_{\mu} = C[x]/I_{\mu} \]

For an ideal with a relatively explicit description. They manage to give generators, and finally they proof with more elementary methods that the Frobenius series is the cocharge invariant

\[ F_{R_{\mu}}(x;t) = t^{n(\mu)}H_{\mu}(x;t^{-1}) = \sum_{\lambda} \tilde{K}_{\lambda\mu}(t)s_{\lambda} \]

where \( \tilde{K}_{\lambda\mu}(t) = t^{n(\mu)}K_{\lambda\mu}(t^{-1}) \) is the cocharge Kostka-Foulkes polynomial.