1.1: Introduction to concept of a limit

Limits

Two key problems led to the initial formulation of calculus:

(1) the tangent problem, or how to determine the slope of a line tangent to a curve at a point;

and (2) the area problem, or how to determine the area under a curve.

The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. We therefore begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach.

Definition (Intuitive): Limit

Let \( f(x) \) be a function defined at all values in an open interval containing \( a \), with the possible exception of \( a \) itself, and let \( L \) be a real number. If all values of the function \( f(x) \) approach the real number \( L \) as the values of \( x \neq a \) approach the number \( a \), then we say that the limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \). (More succinct, as \( x \) gets closer to \( a \), \( f(x) \) gets closer and stays close to \( L \).) Symbolically, we express this idea as

\[
\lim_{x \to a} f(x) = L
\]

We can estimate limits by constructing tables of functional values and by looking at their graphs.
Finding limits numerically:

Example \(\PageIndex{1}\):

Evaluate \(\lim_{x \to 0} x+1\).

Solution

Example \(\PageIndex{2}\):

Evaluate \(\lim_{x \to 0} \frac{\sin x}{x}\) using a table of functional values.

We have calculated the values of \(f(x)=(\sin x)/x\) for the values of \(x\) listed in Table \(\PageIndex{2}\).

**Answer:**

<table>
<thead>
<tr>
<th>(x)</th>
<th>((\sin x)/x)</th>
<th>(x)</th>
<th>((\sin x)/x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1</td>
<td>0.998334166468</td>
<td>0.1</td>
<td>0.998334166468</td>
</tr>
<tr>
<td>-0.01</td>
<td>0.999983333417</td>
<td>0.01</td>
<td>0.999983333417</td>
</tr>
<tr>
<td>-0.001</td>
<td>0.999999983333</td>
<td>0.001</td>
<td>0.999999983333</td>
</tr>
<tr>
<td>-0.0001</td>
<td>0.999999999983</td>
<td>0.0001</td>
<td>0.999999999983</td>
</tr>
</tbody>
</table>

Note: The values in this table were obtained using a calculator and using all the places given in the calculator output.

As we read down each \((\sin x)/x\) column, we see that the values in each column appear to be approaching one. Thus, it is fairly reasonable to conclude that \(\lim_{x\to0} \frac{\sin x}{x}=1\). A calculator-or computer-generated graph of \(f(x)=(\sin x)/x\) would be similar to that shown in Figure \(\PageIndex{2}\), and it confirms our estimate.

*Figure \(\PageIndex{2}\):* The graph of \(f(x)=(\sin x)/x\) confirms the estimate from Table.
The Existence of a Limit

As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the functional values must approach a single real-number value at that point. If the functional values do not approach a single value, then the limit does not exist.

Example \(\PageIndex{3}\): Evaluating a Limit That Fails to Exist

Evaluate \(\displaystyle \lim_{x \to 0} \sin(1/x)\) using a table of values.

**Answer:**

Table \(\PageIndex{3}\) lists values for the function \(\sin(1/x)\) for the given values of \(x\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>(\sin(1/x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1</td>
<td>0.54402110889</td>
</tr>
<tr>
<td>-0.01</td>
<td>0.50636564111</td>
</tr>
<tr>
<td>-0.001</td>
<td>-0.8268795405312</td>
</tr>
<tr>
<td>-0.0001</td>
<td>0.305614388888</td>
</tr>
<tr>
<td>-0.00001</td>
<td>-0.035748797987</td>
</tr>
<tr>
<td>-0.000001</td>
<td>0.349993504187</td>
</tr>
</tbody>
</table>

After examining the table of functional values, we can see that the y-values do not seem to approach any one single value. It appears the limit does not exist. Before drawing this conclusion, let’s take a more systematic approach. Take the following sequence of x-values approaching 0:

\[
\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \frac{2}{9\pi}, \frac{2}{11\pi}, \ldots
\]

The corresponding y-values are

\[
\{1, -1, 1, -1, 1, -1, \ldots\}
\]

At this point we can indeed conclude that \(\displaystyle \lim_{x \to 0} \sin(1/x)\) does not exist. (Mathematicians frequently abbreviate “does not exist” as DNE. Thus, we would write \(\lim_{x \to 0} \sin(1/x)\) DNE.) The graph of \(f(x)=\sin(1/x)\) is shown in Figure \(\PageIndex{6}\) and it gives a clearer picture of the behavior of \(\sin(1/x)\) as \(x\) approaches 0. You can see that \(\sin(1/x)\) oscillates ever more wildly between −1 and 1 as \(x\) approaches 0.
Figure \(\PageIndex{6}\): The graph of \(f(x)=\sin \left(\frac{1}{x}\right)\) oscillates rapidly between \(-1\) and \(1\) as \(x\) approaches 0.

Note

Evaluate \(\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)\) using a table of values.

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