A brief excursion into measure theory

Here we briefly mention some of the basic definitions and results from measure theory, and point out how we used them in the previous lectures. The relevant material is covered in Appendix A.1 in [Dur2010] (Appendix A.1 and A.2 in [Dur2004]). It is not required reading, but if you read it you are perhaps more likely to attain a good understanding of the material that is required...

Definition

i. A \(\pi\)-system is a collection \(\{\cal P\}\) of subsets of a set \(\{\Omega\}\) that is closed under intersection of two sets, i.e., if \(A,B\in\{\cal P\}\) then \(A\cap B\in\{\cal P\}\).

ii. A \(\lambda\)-system is a collection \(\{\cal L\}\) of subsets of a set \(\{\Omega\}\) such that: 1. \(\Omega\in\{\cal L\}\); 2. If \(A,B\in\{\cal L\}\) and \(A\subseteq B\) then \(B\setminus A\in\{\cal L\}\); 3. If \((A_n)_{n=1}^{\infty}\) are all in \(\{\cal L\}\) and \((A_n\uparrow A)\) then \(A\in\{\cal L\}\).

The following is a somewhat technical result that turns out to be quite useful:

Theorem: Dynkin’s \(\pi\)-\(\lambda\) theorem

If \(\{\cal P\}\) is a \(\pi\)-system and \(\{\cal L\}\) is a \(\lambda\)-system that contains \(\{\cal P\}\) then \(\sigma(\{\cal P\})\subset\{\cal L\}\).

\begin{lemma}[Uniqueness theorem] If the values of two probability measures \(\mu_1\) and \(\mu_2\) coincide on a collection \(\{\cal P\}\) of sets, and \(\{\cal P\}\) is a \(\pi\)-system (closed under finite intersection), then \(\mu_1\) and \(\mu_2\) coincide on the generated \(\lambda\)-algebra \(\sigma(\{\cal P\})\).
\end{lemma}
The uniqueness theorem implies for example that to check if random variables \((X)\) and \((Y)\) are equal in distribution, it is enough to check that they have the same distribution functions.

Both of the above results are used in the proof of the following important theorem in measure theory:

\begin{thm}[Carathéodory's extension theorem]
Let \(\mu\) be an almost-probability-measure defined on an algebra \(\{\mathcal{A}\}\) of subsets of a set \(\Omega\). That is, it satisfies all the axioms of a probability measure except it is defined on an algebra and not a \(\sigma\)-algebra; \(\sigma\)-additivity is satisfied whenever the countable union of disjoint sets in the algebra is also an element of the algebra. Then \(\mu\) has a unique extension to a probability measure on the \(\sigma\)-algebra \(\sigma(\{\mathcal{A}\})\) generated by \(\{\mathcal{A}\}\).
\end{thm}

Carathéodory's extension theorem is the main tool used in measure theory for constructing measures: one always starts out by defining the measure on some relatively small family of sets and then extending to the generated \(\sigma\)-algebra (after verifying \(\sigma\)-additivity, which often requires using topological arguments, e.g., involving compactness). Applications include:

1. Existence and uniqueness of Lebesgue measure in \(((0,1)), \(\mathbb{R}\) and \(\mathbb{R}^d\).
2. Existence and uniqueness of probability measures associated with a given distribution function in \(\mathbb{R}\) (sometimes called \(\text{``Lebesgue-Stieltjes measures in } \mathbb{R}\text{''}\)). We proved existence instead by starting with Lebesgue measure and using quantile functions.
3. Existence and uniqueness of Lebesgue-Stieltjes measures in \(\mathbb{R}^d\) (i.e., measures associated with a \(d\)-dimensional joint distribution function). Here there is no good concept analogous to quantile functions, although there are other ways to construct such measures explicitly using ordinary Lebesgue measures.
4. Product measures -- this corresponds to probabilistic experiments which consist of several independent smaller experiments.

Note that Durrett's book also talks about measures that are not probability measures, i.e., the total measure of the space is not 1 and may even be infinite. In this setting, the theorems above can be formulated in greater generality.

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