7.5: Sum-to-Product and Product-to-Sum Formulas

Learning Objectives

- Express products as sums.
- Express sums as products.

A band marches down the field creating an amazing sound that bolsters the crowd. That sound travels as a wave that can be interpreted using trigonometric functions.

For example, Figure \(\PageIndex{2}\) represents a sound wave for the musical note A. In this section, we will investigate trigonometric identities that are the foundation of everyday phenomena such as sound waves.
Expressing Products as Sums

We have already learned a number of formulas useful for expanding or simplifying trigonometric expressions, but sometimes we may need to express the product of cosine and sine as a sum. We can use the **product-to-sum formulas**, which express products of trigonometric functions as sums. Let’s investigate the cosine identity first and then the sine identity.

### Expressing Products as Sums for Cosine

We can derive the product-to-sum formula from the sum and difference identities for cosine. If we add the two equations, we get:

\[
\begin{align*}
\cos \alpha \cos \beta + \sin \alpha \sin \beta &= \cos(\alpha - \beta) \\
\underline{\cos \alpha \cos \beta - \sin \alpha \sin \beta} &= \underline{\cos(\alpha + \beta)} \\
2 \cos \alpha \cos \beta &= \cos(\alpha - \beta) + \cos(\alpha + \beta)
\end{align*}
\]

Then, we divide by 2 to isolate the product of cosines:

\[
\cos \alpha \cos \beta = \frac{1}{2} \left( \cos(\alpha - \beta) + \cos(\alpha + \beta) \right) \tag{eq1}
\]

How to: Given a product of cosines, express as a sum

1. Write the formula for the product of cosines.
2. Substitute the given angles into the formula.

**Example \(\PageIndex{1}\): Writing the Product as a Sum Using the Product-to-Sum Formula for Cosine**

Write the following product of cosines as a sum: \(2\cos\left(\dfrac{7x}{2}\right) \cos\left(\dfrac{3x}{2}\right)\).

**Solution**

We begin by writing the formula for the product of cosines (Equation \ref{eq1}):

\[
\cos \alpha \cos \beta = \frac{1}{2} \left( \cos(\alpha - \beta) + \cos(\alpha + \beta) \right) \nonumber
\]

We can then substitute the given angles into the formula and simplify.
\begin{align*}
2 \cos\left(\dfrac{7x}{2}\right)\cos\left(\dfrac{3x}{2}\right)&= 2\left(\dfrac{1}{2}\right)\cos\left(\dfrac{4x}{2}\right)+\cos\left(\dfrac{10x}{2}\right) \\
&= \cos 2x+\cos 5x
\end{align*}

Exercise \(\PageIndex{1}\)

Use the product-to-sum formula (Equation \ref{eq1}) to write the product as a sum or difference: \(\cos(2\theta)\cos(4\theta)\).

Answer
\[
\dfrac{1}{2}(\cos 6\theta+\cos 2\theta)
\]

Expressing the Product of Sine and Cosine as a Sum

Next, we will derive the product-to-sum formula for sine and cosine from the sum and difference formulas for sine. If we add the sum and difference identities, we get:

\[
\begin{align*}
\cos \alpha \cos \beta+\sin \alpha \sin \beta&= \cos(\alpha-\beta)\\text{[4pt]} \underline{+ \cos \alpha \cos \beta-\sin \alpha \sin \beta}= \cos(\alpha+\beta)\\text{[4pt]} 2 \cos \alpha \cos \beta= \cos(\alpha-\beta)+\cos(\alpha+\beta)\\text{[4pt]} \\
\text{Then, we divide by 2 to isolate the product of cosines:}\ \cos \alpha \cos \beta&= \dfrac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)\text{[4pt]}
\end{align*}
\]

Example \(\PageIndex{2}\): Writing the Product as a Sum Containing only Sine or Cosine

Express the following product as a sum containing only sine or cosine and no products: \(\sin(4\theta)\cos(2\theta)\).

Solution

Write the formula for the product of sine and cosine. Then substitute the given values into the formula and simplify.

\[
\begin{align*}
\sin \alpha \cos \beta&= \dfrac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)\text{[4pt]} \sin(4\theta)\cos(2\theta)&= \dfrac{1}{2}[\sin(6\theta)+\sin(2\theta)\text{[4pt]}
\end{align*}
\]

Exercise \(\PageIndex{2}\)

Use the product-to-sum formula to write the product as a sum: \(\sin(x+y)\cos(x-y)\).

Answer

\[
\dfrac{1}{2}(\sin 2x+\sin 2y)
\]
Expressing Products of Sines in Terms of Cosine

Expressing the product of sines in terms of cosine is also derived from the sum and difference identities for cosine. In this case, we will first subtract the two cosine formulas:

\[
\begin{align*}
\cos(\alpha-\beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\
\underline{-\cos(\alpha+\beta)} &= -(\cos \alpha \cos \beta - \sin \alpha \sin \beta) \\
\cos(\alpha-\beta) - \cos(\alpha+\beta) &= 2 \sin \alpha \sin \beta \\
\text{Then, we divide by 2 to isolate the product of sines:} & \\
\sin \alpha \sin \beta &= \frac{1}{2} \left[ \cos(\alpha-\beta) - \cos(\alpha+\beta) \right]
\end{align*}
\]

Similarly, we could express the product of cosines in terms of sine or derive other product-to-sum formulas.

THE PRODUCT-TO-SUM FORMULAS

The product-to-sum formulas are as follows:

\[
\begin{align*}
\cos \alpha \cos \beta &= \frac{1}{2} \left[ \cos(\alpha-\beta) + \cos(\alpha+\beta) \right] \\
\sin \alpha \cos \beta &= \frac{1}{2} \left[ \sin(\alpha+\beta) + \sin(\alpha-\beta) \right] \\
\sin \alpha \sin \beta &= \frac{1}{2} \left[ \cos(\alpha-\beta) - \cos(\alpha+\beta) \right] \\
\cos \alpha \sin \beta &= \frac{1}{2} \left[ \sin(\alpha+\beta) - \sin(\alpha-\beta) \right]
\end{align*}
\]

Example \(\PageIndex{3}\): Express the Product as a Sum or Difference

Write \(\cos(3\theta) \cos(5\theta)\) as a sum or difference.

Solution

We have the product of cosines, so we begin by writing the related formula. Then we substitute the given angles and simplify.

\[
\begin{align*}
\cos \alpha \cos \beta &= \frac{1}{2} \left[ \cos(\alpha-\beta) + \cos(\alpha+\beta) \right] \\
\cos(3\theta) \cos(5\theta) &= \frac{1}{2} \left[ \cos(3\theta-5\theta) + \cos(3\theta+5\theta) \right] \\
&= \frac{1}{2} \left[ \cos(2\theta) + \cos(8\theta) \right] \quad \text{Use even-odd identity}
\end{align*}
\]

Exercise \(\PageIndex{3}\)

Use the product-to-sum formula to evaluate \(\cos \frac{11\pi}{12} \cos \frac{\pi}{12}\).

Answer

\(\frac{-2 - \sqrt{3}}{4}\)
Expressing Sums as Products

Some problems require the reverse of the process we just used. The sum-to-product formulas allow us to express sums of sine or cosine as products. These formulas can be derived from the product-to-sum identities. For example, with a few substitutions, we can derive the sum-to-product identity for sine. Let \( \dfrac{u+v}{2} = \alpha \) and \( \dfrac{u-v}{2} = \beta \).

Then,

\[
\begin{align*}
\alpha + \beta &= \dfrac{u+v}{2} + \dfrac{u-v}{2} \\
&= \dfrac{2u}{2} \\
&= u \\
\alpha - \beta &= \dfrac{u+v}{2} - \dfrac{u-v}{2} \\
&= \dfrac{2v}{2} \\
&= v
\end{align*}
\]

Thus, replacing \( \alpha \) and \( \beta \) in the product-to-sum formula with the substitute expressions, we have

\[
\begin{align*}
\sin \alpha \cos \beta &= \dfrac{1}{2} [\sin (\alpha + \beta) + \sin (\alpha - \beta)] \\
\sin \left( \dfrac{u+v}{2} \right) \cos \left( \dfrac{u-v}{2} \right) &= \dfrac{1}{2} [\sin u + \sin v] \quad \text{(Substitute for } (\alpha + \beta) \text{ and } (\alpha - \beta)\text{)} \\
2 \sin \left( \dfrac{u+v}{2} \right) \cos \left( \dfrac{u-v}{2} \right) &= \sin u + \sin v
\end{align*}
\]

The other sum-to-product identities are derived similarly.

SUM-TO-PRODUCT FORMULAS

The sum-to-product formulas are as follows:

\[
\begin{align*}
\sin \alpha + \sin \beta &= 2 \sin \left( \dfrac{\alpha + \beta}{2} \right) \cos \left( \dfrac{\alpha - \beta}{2} \right) \\
\sin \alpha - \sin \beta &= 2 \sin \left( \dfrac{\alpha - \beta}{2} \right) \cos \left( \dfrac{\alpha + \beta}{2} \right) \\
\cos \alpha - \cos \beta &= -2 \sin \left( \dfrac{\alpha + \beta}{2} \right) \sin \left( \dfrac{\alpha - \beta}{2} \right) \\
\cos \alpha + \cos \beta &= 2 \sin \left( \dfrac{\alpha + \beta}{2} \right) \sin \left( \dfrac{\alpha - \beta}{2} \right)
\end{align*}
\]

Example \( \PageIndex{4} \): Writing the Difference of Sines as a Product

Write the following difference of sines expression as a product: \( \sin (4\theta) - \sin (2\theta) \).

Solution

We begin by writing the formula for the difference of sines.

\[
\begin{align*}
\sin \alpha - \sin \beta &= 2 \sin \left( \dfrac{\alpha - \beta}{2} \right) \cos \left( \dfrac{\alpha + \beta}{2} \right) \\
&= 2 \sin \left( \dfrac{2\theta}{2} \right) \cos \left( \dfrac{6\theta}{2} \right) \\
&= 2 \sin \theta \cos (3\theta)
\end{align*}
\]
Exercise \(\PageIndex{4}\)

Use the sum-to-product formula to write the sum as a product: \(\sin(3\theta)+\sin(\theta))\).

**Answer**

\[2\sin(2\theta)\cos(\theta)\]

Example \(\PageIndex{5}\): Evaluating Using the Sum-to-Product Formula

Evaluate \(\cos(15°)−\cos(75°))\). Check the answer with a graphing calculator.

**Solution**

We begin by writing the formula for the difference of cosines.

\[
\begin{align*}
\cos \alpha-\cos \beta &= -2 \sin\left(\dfrac{\alpha+\beta}{2}\right) \sin\left(\dfrac{\alpha-\beta}{2}\right) \\
\text{Then we substitute the given angles and simplify.} &\ \\
\cos(15^\circ)-\cos(75^\circ) &= -2\sin\left(\dfrac{15^\circ+75^\circ}{2}\right) \sin\left(\dfrac{15^\circ-75^\circ}{2}\right) \\
&= -2\sin(45^\circ) \sin(-30^\circ) \\
&= -2\left(\dfrac{\sqrt{2}}{2}\right)\left(-\dfrac{1}{2}\right) \\
&= \dfrac{\sqrt{2}}{2}
\end{align*}
\]

Example \(\PageIndex{6}\): Proving an Identity

Prove the identity:

\[
\dfrac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)}=−\tan t
\]

**Solution**

We will start with the left side, the more complicated side of the equation, and rewrite the expression until it matches the right side.

\[
\begin{align*}
\dfrac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)} &= \dfrac{-2 \sin\left(\dfrac{4t+2t}{2}\right) \sin\left(\dfrac{4t-2t}{2}\right) \sin\left(\dfrac{4t+2t}{2}\right) \cos\left(\dfrac{4t-2t}{2}\right) &\ \\
&= \dfrac{-2 \sin\left(\sqrt{2}\right) \left(\dfrac{1}{2}\right) \left(\sqrt{2}\right) \cos\left(\dfrac{1}{2}\right) &\ \\
&= -\dfrac{1}{2}
\end{align*}
\]

**Analysis**

Recall that verifying trigonometric identities has its own set of rules. The procedures for solving an equation are not the same as the procedures for verifying an identity. When we prove an identity, we pick one side to work on and make substitutions
until that side is transformed into the other side.

Example $\PageIndex{7}$: Verifying the Identity Using Double-Angle Formulas and Reciprocal Identities

Verify the identity $({\csc}^2 \theta−2=\cos(2\theta)\sin2\theta)$.

**Solution**

For verifying this equation, we are bringing together several of the identities. We will use the double-angle formula and the reciprocal identities. We will work with the right side of the equation and rewrite it until it matches the left side.

\[
\begin{align*}
\cos(2\theta)\sin2\theta &= \dfrac{1-2 \sin^2 \theta}{\sin^2 \theta} \\
&= \dfrac{1}{\sin^2 \theta} - \dfrac{2 \sin^2 \theta}{\sin^2 \theta} \\
&= {\csc^2 \theta} - 2
\end{align*}
\]

Exercise $\PageIndex{5}$

Verify the identity $(\tan \theta \cot \theta−\cos^2 \theta=\sin^2 \theta)$.

**Answer**

\[
\begin{align*}
\tan \theta \cot \theta - \cos^2 \theta &= \left(\dfrac{\sin \theta}{\cos \theta}\right)\left(\dfrac{\cos \theta}{\sin \theta}\right) - \cos^2 \theta \\
&= 1 - \cos^2 \theta \\
&= \sin^2 \theta
\end{align*}
\]

Media

Access these online resources for additional instruction and practice with the product-to-sum and sum-to-product identities.

- Sum to Product Identities
- Sum to Product and Product to Sum Identities

**Key Equations**

**Product-to-sum Formulas**

\[
\begin{align*}
\cos \alpha \cos \beta &= \dfrac{1}{2}[\cos(\alpha−\beta)+\cos(\alpha+\beta)] \nonumber \\
\sin \alpha \cos \beta &= \dfrac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha−\beta)] \nonumber \\
\sin \alpha \sin \beta &= \dfrac{1}{2}[\cos(\alpha−\beta)−\cos(\alpha+\beta)] \nonumber \\
\cos \alpha \sin \beta &= \dfrac{1}{2}[\sin(\alpha+\beta)−\sin(\alpha−\beta)] \nonumber
\end{align*}
\]

**Sum-to-product Formulas**

\[
\begin{align*}
\sin \alpha + \sin \beta &= 2 \sin(\dfrac{1}{2}\alpha+\beta)\cos(\dfrac{1}{2}\alpha−\beta) \nonumber \\
\sin \alpha + \sin \beta &= 2 \sin(\dfrac{1}{2}\alpha+\beta)\cos(\dfrac{1}{2}\alpha−\beta) \nonumber
\end{align*}
\]
\[
\sin \alpha - \sin \beta = 2 \sin \left( \frac{\alpha - \beta}{2} \right) \cos \left( \frac{\alpha + \beta}{2} \right) 
\]

\[
\cos \alpha - \cos \beta = -2 \sin \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\alpha - \beta}{2} \right) 
\]

\[
\cos \alpha + \cos \beta = 2 \sin \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\alpha - \beta}{2} \right) 
\]

**Key Concepts**

- From the sum and difference identities, we can derive the product-to-sum formulas and the sum-to-product formulas for sine and cosine.
- We can use the product-to-sum formulas to rewrite products of sines, products of cosines, and products of sine and cosine as sums or differences of sines and cosines. See Example \(\PageIndex{1}\), Example \(\PageIndex{2}\), and Example \(\PageIndex{3}\).
- We can also derive the sum-to-product identities from the product-to-sum identities using substitution.
- We can use the sum-to-product formulas to rewrite sum or difference of sines, cosines, or products sine and cosine as products of sines and cosines. See Example \(\PageIndex{4}\).
- Trigonometric expressions are often simpler to evaluate using the formulas. See Example \(\PageIndex{5}\).
- The identities can be verified using other formulas or by converting the expressions to sines and cosines. To verify an identity, we choose the more complicated side of the equals sign and rewrite it until it is transformed into the other side. See Example \(\PageIndex{6}\) and Example \(\PageIndex{7}\).

**Contributors**

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