2.8: Linear Inequalities and Absolute Value Inequalities

Learning Objectives

- Use interval notation.
- Use properties of inequalities.
- Solve inequalities in one variable algebraically.
- Solve absolute value inequalities.

It is not easy to make the honor role at most top universities. Suppose students were required to carry a course load of at least \(12\) credit hours and maintain a grade point average of \(3.5\) or above. How could these honor roll requirements be expressed mathematically? In this section, we will explore various ways to express different sets of numbers, inequalities, and absolute value inequalities.
Using Interval Notation

Indicating the solution to an inequality such as \(x \geq 4\) can be achieved in several ways.

- We can use a number line as shown in Figure \(\PageIndex{2}\). The blue ray begins at \(x = 4\) and, as indicated by the arrowhead, continues to infinity, which illustrates that the solution set includes all real numbers greater than or equal to \(4\).
- We can use **set-builder notation**: \(\{x|x \geq 4\}\), which translates to “all real numbers \(x\) such that \(x\) is greater than or equal to \(4\).” Notice that braces are used to indicate a set.
- The third method is **interval notation**, in which solution sets are indicated with parentheses or brackets. The solutions to \(x \geq 4\) are represented as \([4, \infty)\). This is perhaps the most useful method, as it applies to concepts studied later in this course and to other higher-level math courses.

![Figure \(\PageIndex{2}\)](image)

The main concept to remember is that parentheses represent solutions greater or less than the number, and brackets represent solutions that are greater than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be “equaled.” A few examples of an interval, or a set of numbers in which a solution falls, are \((-\infty, 6)\), or all numbers between \((-\infty, 6)\), including \((-\infty, 6)\), but not including \((-\infty, 6)\); \((-\infty, 0)\), all real numbers between, but not including \((-\infty, 0);\) and \((-\infty, 1)\), all real numbers less than and including \((-\infty, 1)\). Table \(\PageIndex{1}\) outlines the possibilities.

<table>
<thead>
<tr>
<th>Set Indicated</th>
<th>Set-Builder Notation</th>
<th>Interval Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>All real numbers between ((-\infty, b)), but not including ((-\infty, a)) or ((-\infty, b))</td>
<td>((-\infty, a))</td>
<td>((-\infty, a))</td>
</tr>
<tr>
<td>All real numbers greater than ((-\infty, a)), but not including ((-\infty, a))</td>
<td>((-\infty, a))</td>
<td>((-\infty, a))</td>
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<tr>
<td>All real numbers less than ((-\infty, b)), but not including ((-\infty, b))</td>
<td>((-\infty, b))</td>
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</tr>
<tr>
<td>All real numbers greater than ((-\infty, a)), including ((-\infty, a))</td>
<td>((-\infty, a))</td>
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<tr>
<td>Set Indicated</td>
<td>Set-Builder Notation</td>
<td>Interval Notation</td>
</tr>
<tr>
<td>------------------------------------------------------------------------------</td>
<td>--------------------------------------------------------------------------------------</td>
<td>----------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>All real numbers between (a) and (b), including (a) and (b)</td>
<td>({x</td>
<td>a \leq x \leq b})</td>
</tr>
<tr>
<td>All real numbers less than (a) or greater than (b)</td>
<td>({x</td>
<td>x &lt; a \text{ or } x &gt; b})</td>
</tr>
<tr>
<td>All real numbers</td>
<td>({x</td>
<td>x \text{ is all real space numbers}})</td>
</tr>
</tbody>
</table>

Example \(\PageIndex{1}\): Using Interval Notation to Express All Real Numbers Greater Than or Equal to a

Use interval notation to indicate all real numbers greater than or equal to \((-2)\).

Solution

Use a bracket on the left of \((-2)\) and parentheses after infinity: \((-2,\infty)\). The bracket indicates that \((-2)\) is included in the set with all real numbers greater than \((-2)\) to infinity.

Exercise \(\PageIndex{1}\)

Use interval notation to indicate all real numbers between and including \((-3)\) and \((5)\).

Answer

\([[-3,5]]\)

Example \(\PageIndex{2}\): Using Interval Notation to Express All Real Numbers Less Than or Equal to a or Greater Than or Equal to b

Write the interval expressing all real numbers less than or equal to \((-1)\) or greater than or equal to \((1)\).

Solution

We have to write two intervals for this example. The first interval must indicate all real numbers less than or equal to \((1)\). So, this interval begins at \((−\infty)\) and ends at \((-1)\), which is written as \((−\infty,-1]\)..

The second interval must show all real numbers greater than or equal to \((1)\), which is written as \((1,\infty]\)). However, we want to combine these two sets. We accomplish this by inserting the union symbol, \(\cup\), between the two intervals.

\([[-\infty,-1]\cup[1,\infty)]\)

Exercise \(\PageIndex{2}\)

Express all real numbers less than \((-2)\) or greater than or equal to \((3)\) in interval notation.
Using the Properties of Inequalities

When we work with inequalities, we can usually treat them similarly to but not exactly as we treat equalities. We can use the addition property and the multiplication property to help us solve them. The one exception is when we multiply or divide by a negative number; doing so reverses the inequality symbol.

**PROPERTIES OF INEQUALITIES**

**Addition Property**
- If \(a<b\), then \((a+c)<(b+c)\).

**Multiplication Property**
- If \(a<b\) and \((c>0)\), then \((ac)<(bc)\).
- If \(a<b\) and \((c<0)\), then \((ac)>(bc)\).

These properties also apply to \((a\leq b)\), \((a>b)\), and \((a\geq b)\).

**Example \(\PageIndex{3}\): Demonstrating the Addition Property**

Illustrate the addition property for inequalities by solving each of the following:

a. \((a)\ (x-15<4)\)

\[
\begin{align*}
\quad x-15 &< 4 \\
\quad x-15+15 &< 4+15 \\
\quad x &< 19
\end{align*}
\]

b. \((b)\ (6\geq x-1)\)

\[
\begin{align*}
\quad 6+1 &\geq x-1+1 \\
\quad 7 &\geq x
\end{align*}
\]

c. \((c)\ (x+7>9)\)

\[
\begin{align*}
\quad \text{Solution}
\end{align*}
\]

The addition property for inequalities states that if an inequality exists, adding or subtracting the same number on both sides does not change the inequality.

a.

\[
\begin{align*}
\quad \text{solution:} \\
\quad x-15 &< 4 & x-15+15 &< 4+15 & x &< 19
\end{align*}
\]

b.

\[
\begin{align*}
\quad \text{solution:} \\
\quad 6 &\geq x-1 & 6+1 &\geq x-1+1 & 7 &\geq x
\end{align*}
\]

c.
\[
\begin{align*}
  x+7 &> 9 \\
  x+7-7 &> 9-7 \\
  x &> 2
\end{align*}
\]

Exercise \(\PageIndex{3}\)

Solve: \((3x-2<1)\).

Answer

\(x<1\)

Example \(\PageIndex{4}\): Demonstrating the Multiplication Property

Illustrate the multiplication property for inequalities by solving each of the following:

a. \((3x<6)\)

b. \((-2x+1\geq 5)\)

c. \((5-x>10)\)

Solution

a.

\[
\begin{align*}
  3x &< 6 \\
  \frac{1}{3}(3x) &< (6)\frac{1}{3} \\
  x &< 2
\end{align*}
\]

b.

\[
\begin{align*}
  -2x-1 &\geq 5 \\
  -2x &\geq 6 \\
  \left(-\frac{1}{2}\right)(-2) &\geq (6)\left(-\frac{1}{2}\right) \text{ Multiply by } \left(-\frac{1}{2}\right) \\
  x &\leq -3 \text{ Reverse the inequality.}
\end{align*}
\]

c.

\[
\begin{align*}
  5-x &> 10 \\
  -x &> 5 \\
  (-1)(-x) &> (5)(-1) \text{ Multiply by } -1 \text{ Reverse the inequality.}
\end{align*}
\]

Exercise \(\PageIndex{4}\)

Solve: \((4x+7\geq 2x-3)\).

Answer

\(x\geq -5\)

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**Solving Inequalities in One Variable Algebraically**

As the examples have shown, we can perform the same operations on both sides of an inequality, just as we do with equations;
we combine like terms and perform operations. To solve, we isolate the variable.

Example \(\PageIndex{5}\): Solving an Inequality Algebraically

Solve the inequality: \((13−7x≥10x−4)\).

**Solution**

Solving this inequality is similar to solving an equation up until the last step.

\[
\begin{align*}
13-7x&\geq 10x-4 \\
13-17x&\geq -4 & \text{Move variable terms to one side of the inequality} \\
-17x&\geq -17 & \text{Isolate the variable term} \\
x&\leq 1 & \text{Dividing both sides by -17 reverses the inequality.}
\end{align*}
\]

The solution set is given by the interval \((−\infty,1]\), or all real numbers less than and including \((1)\).

Exercise \(\PageIndex{5}\)

Solve the inequality and write the answer using interval notation: \((-x+4<\dfrac{1}{2}x+1)\).

**Answer**

\([(2,\infty))\]

Example \(\PageIndex{6}\): Solving an Inequality with Fractions

Solve the following inequality and write the answer in interval notation: \((-\dfrac{3}{4}x≥-\dfrac{5}{8}+\dfrac{2}{3}x)\).

**Solution**

We begin solving in the same way we do when solving an equation.

\[
\begin{align*}
-\dfrac{3}{4}x&\geq -\dfrac{5}{8}+\dfrac{2}{3}x \\
-\dfrac{3}{4}x-\dfrac{2}{3}x&\geq -\dfrac{5}{8} & \text{Put variable terms on one side.} \\
-\dfrac{9}{12}x-\dfrac{8}{12}x&\geq -\dfrac{5}{8} & \text{Write fractions with common denominator.} \\
-\dfrac{17}{12}x&\geq -\dfrac{15}{34} & \text{Multiplying by a negative number reverses the inequality.}
\end{align*}
\]

The solution set is the interval \((\left(-\infty,-\dfrac{15}{34}\right])\).

Exercise \(\PageIndex{6}\)

Solve the inequality and write the answer in interval notation: \((-\dfrac{5}{6}x≤\dfrac{3}{4}+\dfrac{8}{3}x)\).
\( \left[ -\frac{3}{14}, \infty \right) \)

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