8.4E: Exercises for the Logistic Equation

**Basic Logistic Model**

For problems 1 - 11, consider the logistic equation in the form \( P' = CP - P^2 \). Draw the directional field and find the stability of the equilibria.

1) \( C=3 \)

2) \( C=0 \)

**Answer:**

\( P=0 \) semi-stable
3) \( C = -3 \)

4) Solve the logistic equation for \( C = 10 \) and an initial condition of \( P(0) = 2 \).

Answer:
\[
P = \frac{10e^{10x}}{e^{10x} + 4}
\]

5) Solve the logistic equation for \( C = -10 \) and an initial condition of \( P(0) = 2 \).

6) A population of deer inside a park has a carrying capacity of \( 200 \) and a growth rate of \( 2\% \). If the initial population is \( 50 \) deer, what is the population of deer at any given time?

Answer:
\[
P(t) = \frac{10000e^{0.02t}}{150 + 50e^{0.02t}}
\]

7) A population of frogs in a pond has a growth rate of \( 5\% \). If the initial population is \( 1000 \) frogs and the carrying capacity is \( 6000 \), what is the population of frogs at any given time?

8) [T] Bacteria grow at a rate of \( 20\% \) per hour in a petri dish. If there is initially one bacterium and a carrying capacity of \( 1 \) million cells, how long does it take to reach \( 500,000 \) cells?
9) [T] Rabbits in a park have an initial population of \(10\) and grow at a rate of \(4\%\) per year. If the carrying capacity is \(500\), at what time does the population reach \(100\) rabbits?

Answer:
\(69\) hours \(5\) minutes

10) [T] Two monkeys are placed on an island. After \(5\) years, there are \(8\) monkeys, and the estimated carrying capacity is \(25\) monkeys. When does the population of monkeys reach \(16\) monkeys?

Answer:
\(8\) years \(11\) months

11) [T] A butterfly sanctuary is built that can hold \(2000\) butterflies, and \(400\) butterflies are initially moved in. If after \(2\) months there are now \(800\) butterflies, when does the population get to \(1500\) butterflies?

Logistic Population Model with Depletion

The following problems consider the logistic equation with an added term for depletion, either through death or emigration.

12) [T] The population of trout in a pond is given by \(P'=0.4P\left(1-\frac{P}{10000}\right)-400\), where \(400\) trout are caught per year. Use your calculator or computer software to draw a directional field and draw a few sample solutions. What do you expect for the behavior?
13) In the preceding problem, what are the stabilities of the equilibria \((0<P_1<P_2)\)?

14) [T] For the preceding problem, use software to generate a directional field for the value \((f=400)\). What are the stabilities of the equilibria?

**Answer:**

\((P_1)\) semi-stable
15) [T] For the preceding problems, use software to generate a directional field for the value $f=600.$ What are the stabilities of the equilibria?

16) [T] For the preceding problems, consider the case where a certain number of fish are added to the pond, or $f=-200.$ What are the nonnegative equilibria and their stabilities?

Answer:

$(P_2>0)$ stable
It is more likely that the amount of fishing is governed by the current number of fish present, so instead of a constant number of fish being caught, the rate is proportional to the current number of fish present, with proportionality constant \( k \), as

\[
P' = 0.4P \left(1 - \frac{P}{10000}\right) - kP.
\]

17) [T] For the previous fishing problem, draw a directional field assuming \( k=0.1 \). Draw some solutions that exhibit this behavior. What are the equilibria and what are their stabilities?

18) [T] Use software or a calculator to draw directional fields for \( k=0.4 \). What are the nonnegative equilibria and their stabilities?

**Answer:**

\( P_1=0 \) is semi-stable
19) [T] Use software or a calculator to draw directional fields for \( k=0.6 \). What are the equilibria and their stabilities?

20) Solve this equation, assuming a value of \( k=0.05 \) and an initial condition of \( 2000 \) fish.

Answer:
\[
y = \frac{-20}{4 \times 10^{-6} - 0.002e^{0.01t}}
\]

21) Solve this equation, assuming a value of \( k=0.05 \) and an initial condition of \( 5000 \) fish.

---

**Minimal Sustainable Population Thresholds**

The following problems add in a minimal threshold value for the species to survive, \( T \), which changes the differential equation to \( P'(t) = rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{T}{P}\right) \).

22) Draw the directional field of the threshold logistic equation, assuming \( K=10, r=0.1, T=2 \). When does the population survive? When does it go extinct?
23) For the preceding problem, solve the logistic threshold equation, assuming the initial condition \( P(0)=P_0 \).

24) Bengal tigers in a conservation park have a carrying capacity of \( 100 \) and need a minimum of \( 10 \) to survive. If they grow in population at a rate of \( 1\% \) per year, with an initial population of \( 15 \) tigers, solve for the number of tigers present.

Answer:
\[
P(t)=\frac{850+500e^{0.009t}}{85+5e^{0.009t}}
\]

25) A forest containing ring-tailed lemurs in Madagascar has the potential to support \( 5000 \) individuals, and the lemur population grows at a rate of \( 5\% \) per year. A minimum of 500 individuals is needed for the lemurs to survive. Given an initial population of \( 600 \) lemurs, solve for the population of lemurs.

26) The population of mountain lions in Northern Arizona has an estimated carrying capacity of \( 250 \) and grows at a rate of \( 0.25\% \) per year and there must be \( 25 \) for the population to survive. With an initial population of \( 30 \) mountain lions, how many years will it take to get the mountain lions off the endangered species list (at least \( 100 \))?

Answer:
\[
13 \text{ years months}
\]

The Gompertz Equation

The following questions consider the Gompertz equation, a modification for logistic growth, which is often used for modeling cancer growth, specifically the number of tumor cells.
27) The Gompertz equation is given by \( P(t)' = α \ln \left( \frac{K}{P(t)} \right) P(t) \). Draw the directional fields for this equation assuming all parameters are positive, and given that \( K = 1 \).

28) Assume that for a population, \( K = 1000 \) and \( α = 0.05 \). Draw the directional field associated with this differential equation and draw a few solutions. What is the behavior of the population?

**Answer:**

![Directional Field](image)

29) Solve the Gompertz equation for generic \( α \) and \( K \) and \( P(0) = P_0 \).

30) [T] The Gompertz equation has been used to model tumor growth in the human body. Starting from one tumor cell on day \( 1 \) and assuming \( α = 0.1 \) and a carrying capacity of \( 10 \) million cells, how long does it take to reach “detection” stage at \( 5 \) million cells?

**Answer:**

\( 31.465 \) days

31) [T] It is estimated that the world human population reached \( 3 \) billion people in \( 1959 \) and \( 6 \) billion in \( 1999 \). Assuming a carrying capacity of \( 16 \) billion humans, write and solve the differential equation for logistic growth, and determine what year the population reached \( 7 \) billion.

32) [T] It is estimated that the world human population reached \( 3 \) billion people in \( 1959 \) and \( 6 \) billion in \( 1999 \). Assuming a carrying capacity of \( 16 \) billion humans, write and solve the differential equation for Gompertz growth, and determine what year the population reached \( 7 \) billion. Was logistic growth or Gompertz growth more accurate, considering world population reached \( 7 \) billion on October \( 31, 2011 \)?
33) Show that the population grows fastest when it reaches half the carrying capacity for the logistic equation \( P'=rP\left(1−\frac{P}{K}\right)\).

34) When does population increase the fastest in the threshold logistic equation \( P'(t)=rP\left(1−\frac{P}{K}\right)\left(1−\frac{T}{P}\right)\)?

\( \frac{K+T}{2} \)

35) When does population increase the fastest for the Gompertz equation \( P(t)'=α\ln\left(\frac{K}{P(t)}\right)P(t)\)?

Below is a table of the populations of whooping cranes in the wild from 1940 to 2000. The population rebounded from near extinction after conservation efforts began. The following problems consider applying population models to fit the data. Assume a carrying capacity of 10,000 cranes. Fit the data assuming years since 1940 (so your initial population at time 0 would be 22 cranes).

<table>
<thead>
<tr>
<th>Year (years since conservation began)</th>
<th>Whooping Crane Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1940(0)</td>
<td>22</td>
</tr>
<tr>
<td>1950(10)</td>
<td>31</td>
</tr>
<tr>
<td>1960(20)</td>
<td>36</td>
</tr>
<tr>
<td>1970(30)</td>
<td>57</td>
</tr>
<tr>
<td>1980(40)</td>
<td>91</td>
</tr>
<tr>
<td>1990(50)</td>
<td>159</td>
</tr>
<tr>
<td>2000(60)</td>
<td>256</td>
</tr>
</tbody>
</table>

Source: https://www.savingcranes.org/images/...we_numbers.pdf

36) Find the equation and parameter \( r \) that best fit the data for the logistic equation.

\( r=0.0405 \)

37) Find the equation and parameters \( r \) and \( T \) that best fit the data for the threshold logistic equation.

38) Find the equation and parameter \( α \) that best fit the data for the Gompertz equation.
39) Graph all three solutions and the data on the same graph. Which model appears to be most accurate?

40) Using the three equations found in the previous problems, estimate the population in 2010 (year 70 after conservation). The real population measured at that time was 437. Which model is most accurate?


Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.