2.6: Powers with Arbitrary Real Exponents. Irrationals

In complete fields, one can define \(a^r\) for any \(a > 0\) and \(r \in E^1\) (for \(r \in N\), see §§5-6, Example (\(\text{(f)}\))). First of all, we have the following theorem.

Theorem \(\PageIndex{1}\)

Given \(a \geq 0\) in a complete field \(F\) and a natural number \(n \in E^1\), there always is a unique element \(p \in F, p \geq 0,\) such that

\[p^n = a.\]

It is called the \(n\)th root of \(a,\) denoted

\[\sqrt[n]{a} \text{ or } a^{1/n}.\]

(\(\text{Note that } \sqrt[n]{a} \geq 0,\) by definition.)

Proof

A direct proof, from the completeness axiom, is sketched in Problems 1 and 2 below. We shall give a simpler proof in Chapter 4,§9, Example (a). At present, we omit it and temporarily take Theorem 1 for granted. Hence we obtain the following result.

Theorem \(\PageIndex{2}\)

Every complete field \(F\) (such as \(E^1\)) has irrational elements, i.e., elements that are not rational.
In particular, \( \sqrt{2} \) is irrational.

**Proof**

By Theorem 1, \((F)\) has the element

\[ p = \sqrt{2} \text{ with } p^2 = 2 \]

Seeking a contradiction, suppose \( \sqrt{2} \) is rational, i.e.,

\[ \sqrt{2} = \frac{m}{n} \]

for some \((m, n) \in \mathbb{N}\) in lowest terms (see §7, final note).

Then \((m)\) and \((n)\) are not both even (otherwise, reduction by 2 would yield a smaller \((n)\)). From \((m / n = \sqrt{2})\) we obtain

\[ m^2 = 2n^2 \]

so \(m^2\) is even.

Only even elements have even squares, however. Thus \((m)\) itself must be even; i.e., \((m = 2r)\) for some \((r) \in \mathbb{N}\). It follows that

\[ 4r^2 = m^2 = 2n^2 \text{ i.e., } 2r^2 = n^2 \]

and, by the same argument, \((n)\) must be even.

This contradicts the fact that \((m)\) and \((n)\) are not both even, and this contradiction shows that \( \sqrt{2} \) must be irrational. \(\square\)

**Note 1.** Similarly, one can prove the irrationality of \( \sqrt{a} \) where \( a \in \mathbb{N} \) and \( a \) is not the square of a natural. See Problem 3 below for a hint.

**Note 2.** Theorem 2 shows that the field \((\mathbb{R})\) of all rationals is not complete (for it contains no irrationals), even though it is Archimedean (see Problem 6). Thus the Archimedean property does not imply completeness (but see Theorem 1 of §10).

Next, we define \( a^r \) for any rational number \( r > 0 \).

**Definition**

Given \((a \geq 0)\) in a complete field \((F, )\) and a rational number

\[ r = \frac{m}{n} \quad \text{left}(m, n \in \mathbb{N} \text{ subseq } \mathbb{E}^+ {1} \text{ right}) \]

we define
Here we must clarify two facts.

(1) If \(n=1,\) we have
\[
[a^r = \sqrt[n]{a^m} = a^{m/1} = a^m.]
\]
If \(m=1,\) we get
\[
[a^r = a^{1/n} = \sqrt[n]{a}.]
\]
Thus Definition 1 agrees with our previous definitions of \(a^m\) and \(\sqrt[n]{a}\) \((m, n \in \mathbb{N}).\)

(2) If \(r\) is written as a fraction in two different ways,
\[
[r=\frac{m}{n} = \frac{p}{q},]
\]
then, as is easily seen,
\[
[\sqrt[n]{a^m} = \sqrt[q]{a^p} = a^r,]
\]
and so our definition is \emph{unambiguous} (independent of the particular representation of \(r\)).

Indeed,
\[
[\frac{m}{n} = \frac{p}{q} \text{ implies } m q = n p,]
\]
whence
\[
[a^{m q} = a^{p n},]
\]
i.e.,
\[
[\left(a^m\right)^q = \left(a^p\right)^n;]
\]
cf. §§5-6, Problem 6.

By definition, however,
\[
[\left(\sqrt[n]{a^m}\right)^n = a^m \text{ and } \left(\sqrt[q]{a^p}\right)^q = a^p.]
\]
Substituting this in \(\left(a^m\right)^q = \left(a^p\right)^n\) we get
\[
[\left(\sqrt[n]{a^m}\right)^{n q} = \left(\sqrt[q]{a^p}\right)^{n q}.]
\]
whence
\[
\sqrt[n]{a^{m}} = \sqrt[q]{a^{p}}.\]

Thus Definition 1 is valid, indeed.

By using the results of Problems 4 and 6 of §§5-6, the reader will easily obtain analogous formulas for powers with positive rational exponents, namely,
\[
\begin{aligned}
\begin{aligned}
& a^{r} a^{s} = a^{r+s} ; (a^{r})^{s} = a^{rs} ; (ab)^{r} = a^{r} b^{r} ; a^{r} < a^{s} \text{ if } 0 < a < 1 \text{ and } r > s \\
& a^{r} < a^{s} \iff a^{r} < b^{r} (a, b > 0) ; a^{r} > a^{s} \text{ if } a > 1 \text{ and } r > s ; 1^{r} = 1
\end{aligned}
\end{aligned}
\]

Henceforth we assume these formulas known, for rational \((r, s > 0)\).

Next, we define \(\langle a^{r} \rangle \) for any real \((r > 0)\) and any element \((a > 1)\) in a complete field \((F,.)\).

Let \(\langle A_{a r} \rangle\) denote the set of all members of \((F)\) of the form \(\langle a^{x} \rangle\) with \((x \in R)\) and \((0 < x \leq r)\); i.e.,
\[
\langle A_{a r} \rangle = \{a^{x} | 0 < x \leq r, x \text{ rational}\}.
\]

By the density of rationals in \((E^1)\) (Theorem 3 of §10), such rationals \((x)\) do exist; thus \(\langle A_{a r} \rangle \neq \emptyset\).

Moreover, \(\langle A_{a r} \rangle\) is right bounded in \((F,.)\) Indeed, fix any rational number \((y > r)\). By the formulas in \((1)\), we have, for any positive rational \((x \leq r)\),
\[
[a^{y}] = a^{x+y-x} = a^{x} a^{y-x} > a^{x}
\]
since \((a > 1)\) and \((y-x > 0)\) implies
\[
[a^{y-x}] > 1.\]

Thus \(\langle a^{y} \rangle \) is an upper bound of all \(\langle a^{x} \rangle\) in \(\langle A_{a r} \rangle\).

Hence, by the assumed completeness of \((F,.)\) sup \(\langle A_{a r} \rangle\) exists. So we may define
\[
\langle a^{r} \rangle = \sup A_{a r}.
\]

We also put
\[
\langle a^{-r} \rangle = \frac{1}{a^{r}}.
\]

If \((0 < a < 1)\) (so that \(\langle a \rangle > 1\)), we put
\[
\langle a^{r} \rangle = \begin{cases}
\langle a \rangle^{r} & \text{if } r < 1 \\
\frac{1}{a^{r}} & \text{if } r > 1
\end{cases}
\]
where
\[
\left(\frac{1}{a}\right)^{r} = \sup A_{1 / a, r},
\]
as above.

Summing up, we have the following definitions.

Definition

Given \(a>0\) in a complete field \(F\), and \(r \in E^{1}\), we define the following.

(i) If \(r>0\) and \(a>1\), then
\[
a^{r} = \sup A_{a r} = \sup \left\{a^{x} \mid 0<x \leq r, x \text{ rational} \right\}
\]
(ii) If \(r>0\) and \(0<a<1\), then \(a^{r} = \frac{1}{(1 / a)^{r}}\), also written \(((1 / a)^{-r} \cdot r)\)
(iii) \(a^{-r} = 1 / a^{r}\). (This defines powers with negative exponents as well.)

We also define \(0^{r} = 0\) for any real \(r>0\), and \(a^{0} = 1\) for any \(a \in F\), \(a \neq 0\); \(0^{0}\) remains undefined.

The power \(a^{r}\) is also defined if \(a<0\) and \(r\) is a rational \(\frac{m}{n}\) with \(n\) because
\[
a^{r} = \sqrt[n]{a^{m}}
\]
has sense in this case. (Why?) This does not work for other values of \(r\). Therefore, in general, we assume \(a>0\).

Again, it is easy to show that the formulas in (1) remain also valid for powers with real exponents (see Problems 8-13 below), provided \(F\) is complete.