13.3: Changing to a Basis of Eigenvectors

If we are changing to a basis of eigenvectors, then there are various simplifications:

1. Since \( L: V \to V \), most likely you already know the matrix \( M \) of \( L \) using the same input basis as output basis \( S=(u_1, \ldots, u_n) \) (say).

2. In the new basis of eigenvectors \( S'(v_1, \ldots, v_n) \), the matrix \( D \) of \( L \) is diagonal because \( Lv_i = \lambda_i v_i \) and so

\[
(L(v_1), L(v_2), \ldots, L(v_n)) = (v_1, v_2, \ldots, v_n) \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \ddots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix}
\]

3. If \( P \) is the change of basis matrix from \( S' \) to \( S \), the diagonal matrix of eigenvalues \( D \) and the original matrix are related by \( D = P^{-1}MP \).

This motivates the following definition:

Definition
A matrix \((M)\) is diagonalizable if there exists an invertible matrix \((P)\) and a diagonal matrix \((D)\) such that

\[
D = P^{-1}MP.
\]

We can summarize as follows:

1. Change of basis rearranges the components of a vector by the change of basis matrix \((P)\), to give components in the new basis.
2. To get the matrix of a linear transformation in the new basis, we \((\text{conjugate})\) the matrix of \((L)\) by the change of basis matrix: \((M \mapsto P^{-1}MP)\).

If for two matrices \((N)\) and \((M)\) there exists a matrix \((P)\) such that \((M = P^{-1}NP)\), then we say that \((M)\) and \((N)\) are \((\text{similar})\). Then the above discussion shows that diagonalizable matrices are similar to diagonal matrices.

**Corollary**

A square matrix \((M)\) is diagonalizable if and only if there exists a basis of eigenvectors for \((M)\). Moreover, these eigenvectors are the columns of the change of basis matrix \((P)\) which diagonalizes \((M)\).

**Example 122**

Let’s try to diagonalize the matrix

\[
M = \begin{pmatrix}
-14 & -28 & -44 \\
-7 & -14 & -23 \\
9 & 18 & 29
\end{pmatrix}
\]

The eigenvalues of \((M)\) are determined by \(\det(M - \lambda I) = -\lambda^3 + \lambda^2 + 2\lambda = 0\).

So the eigenvalues of \((M)\) are \((-1, 0, 2)\), and associated eigenvectors turn out to be

\[
v_{-1} = \begin{pmatrix} -8 \\ -1 \\ 3 \end{pmatrix}, \quad v_{0} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad v_{2} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}
\]

In order for \((M)\) to be diagonalizable, we need the vectors \((v_{-1}, v_{0}, v_{2})\) to be linearly independent. Notice that the matrix

\[
P = \begin{pmatrix}
-8 & -2 & -1 \\
-1 & 1 & -1 \\
3 & 0 & 1
\end{pmatrix}
\]

is diagonalizable.
is invertible because its determinant is \((-1)\). Therefore, the eigenvectors of \((M)\) form a basis of \((\Re)\), and so \((M)\) is diagonalizable.

Moreover, because the columns of \((P)\) are the components of eigenvectors,

$$
MP = \begin{pmatrix} Mv_1 & Mv_2 & Mv_3 \end{pmatrix} = \begin{pmatrix} -1 \cdot v_1 & 0 \cdot v_2 & 2 \cdot v_3 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
$$

Hence, the matrix \((P)\) of eigenvectors is a change of basis matrix that diagonalizes \((M)\):

$$
[P^{-1}]MP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
$$

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