4.1: One-Dimensional Wave Equation

The one-dimensional wave equation is given by

\begin{equation}
\label{waveone}
\frac{1}{c^2}u_{tt} - u_{xx} = 0,
\end{equation}

where $u = u(x,t)$ is a scalar function of two variables and $c$ is a positive constant. According to previous considerations, all $C^2$-solutions of the wave equation are

\begin{equation}
\label{wavegen}
u(x,t) = f(x+ct) + g(x-ct),
\end{equation}

with arbitrary $(C^2)$-functions $f$ and $g$.

The Cauchy initial value problem for the wave equation is to find a $(C^2)$-solution of

\begin{eqnarray*}
\dfrac{1}{c^2}u_{tt} - u_{xx} &=& 0 \\
u(x,0) &=& \alpha(x) \\
u_t(x,0) &=& \beta(x),
\end{eqnarray*}
where \( \alpha, \beta \in C^2(-\infty, \infty) \) are given.

**Theorem 4.1.** There exists a unique \( C^2(\mathbb{R}^1 \times \mathbb{R}^1) \)-solution of the Cauchy initial value problem, and this solution is given by d'Alembert's formula

\[
\begin{equation}
\label{waveform}
u(x,t)=\frac{\alpha(x+ct)+\alpha(x-ct)}{2}+\frac{1}{2c} \int_{x-ct}^{x+ct} \beta(s) \, ds.
\end{equation}
\]

**Proof.** Assume there is a solution \( u(x,t) \) of the Cauchy initial value problem, then it follows from \((\text{ref} \{\text{wavegen}\})\) that

\[
\begin{eqnarray}
\label{ini1} u(x,0)&=&f(x)+g(x)=\alpha(x) \\
\label{ini2} u_t(x,0)&=&cf'(x)-cg'(x)=\beta(x).
\end{eqnarray}
\]

From \((\text{ref} \{\text{ini1}\})\) we obtain

\[
[f(x)+g(x)=\alpha'(x),]
\]

which implies, together with \((\text{ref} \{\text{ini2}\})\), that

\[
\begin{eqnarray*}
\label{12a} f'(x)&=&\frac{\alpha'(x)+\beta(x)/c}{2} \\
\label{12b} g'(x)&=&\frac{\alpha'(x)-\beta(x)/c}{2}.
\end{eqnarray*}
\]

Then

\[
\begin{eqnarray*}
\label{eqnarray*} f(x)&=&\frac{\alpha(x)}{2}+\frac{1}{2c} \int_0^x \beta(s) \, ds +C_1 \\
g(x)&=&\frac{\alpha(x)}{2}-\frac{1}{2c} \int_0^x \beta(s) \, ds +C_2.
\end{eqnarray*}
\]

The constants \( (C_1), (C_2) \) satisfy

\[
[C_1+C_2=f(x)+g(x)-\alpha(x)=0,]
\]

see \((\text{ref} \{\text{ini1}\})\). Thus each \( C^2 \)-solution of the Cauchy initial value problem is given by d'Alembert's formula. On the other hand, the function \( u(x,t) \) defined by the right hand side of \((\text{ref} \{\text{waveform}\})\) is a solution of the initial value problem.
Corollaries. 1. The solution \(u(x,t)\) of the initial value problem depends on the values of \(\alpha\) at the endpoints of the interval \([x-ct,x+ct]\) and on the values of \(\beta\) on this interval only, see Figure 4.1.1. The interval \([x-ct,x+ct]\) is called the domain of dependence.

![Interval of dependence](image1)

**Figure 4.1.1: Interval of dependence**

2. Let \(P\) be a point on the \(x\)-axis. Then we ask which points \((x,t)\) need values of \(\alpha\) or \(\beta\) at \(P\) in order to calculate \(u(x,t)\)? From the d'Alembert formula it follows that this domain is a cone, see Figure 4.2.1. This set is called the domain of influence.

![Domain of influence](image2)

**Figure 4.2.1: Domain of influence**

\(^1\) d'Alembert, Jean Baptiste le Rond, 1717-1783

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