4.E: 4: Hyperbolic Equations (Exercises)

These are homework exercises to accompany Miersemann's "Partial Differential Equations" Textmap. This is a textbook targeted for a one semester first course on differential equations, aimed at engineering students. Partial differential equations are differential equations that contain unknown multivariable functions and their partial derivatives. Prerequisite for the course is the basic calculus sequence.

Q4.1

Show that \((u(x,t)) \in C^2(\mathbb{R}^2)\) is a solution of the one-dimensional wave equation

\[ u_{tt} = c^2 u_{xx} \]

if and only if

\[ u(A) + u(C) = u(B) + u(D) \]

holds for all parallelograms \((ABCD)\) in the \((x,t)\)-plane, which are bounded by characteristic lines, see Figure 4.E.1.

![Figure 4.1: Figure to the exercise](image)

Q4.2: Method of separation of variables

Let \((v_k(x))\) be an eigenfunction to the eigenvalue of the eigenvalue problem

\[ (-v''(x) = \lambda v(x)) \text{ in } (0,1), \quad v(0) = v(1) = 0. \]

and let \((w_k(t))\) be a solution of differential equation.
\[-w''(t)=\lambda_kw(t)\]

Prove that \(v_k(x)w_k(t)\) is a solution of the partial differential equation (wave equation) \(u_{tt}=u_{xx}\).

Q4.3

Solve for given \(f(x)\) and \(\mu\in\mathbb{R}^1\) the initial value problem

\[
\begin{align*}
    u_t + u_x + \mu u_{xxx} &= 0 \quad \text{in} \quad \mathbb{R}^1 \times \mathbb{R}^1_+ \\
    u(x,0) &= f(x)
\end{align*}
\]

Q4.4

Let \(S := \{(x,t); t = \gamma x\}\) be space-like, i.e., \(|\gamma| < 1/c^2\) in \((x,t)\)-space, \((x=(x_1,x_2,x_3))\). Show that the Cauchy initial value problem \(\Box u=0\) with data for \(u\) on \(S\) can be transformed using the Lorentz-transform

\[
\begin{align*}
    x_1 &= \frac{x_1 - \gamma c^2 t}{\sqrt{1-\gamma^2 c^2}} \\
    x_2' &= x_2, \quad x_3' = x_3 \\
    t' &= \frac{t - \gamma x_1}{\sqrt{1-\gamma^2 c^2}}
\end{align*}
\]

into the initial value problem, in new coordinates,

\[
\begin{align*}
    \Box u &= 0 \\
    u(x',0) &= f(x') \\
    u_{t'}(x',0) &= g(x')
\end{align*}
\]

Here we denote the transformed function by \(u'\) again.

Q4.5

(i) Show that

\[ u(x,t):=\sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{\pi n}{l} t\right) \sin\left(\frac{\pi n}{l} x\right) \]

is a \((C^2)\)-solution of the wave equation \(u_{tt}=u_{xx}\) if \(|\alpha_n| \leq c/n^4\), where the constant \(|c|\) is independent of \(|n|\).
(ii) Set
$$\alpha_n := \int_0^l f(x) \sin \left( \frac{\pi n}{l} x \right) \, dx.$$ 
Prove \(|\alpha_n| \leq c/n^4\), provided \(f \in C^4_0(0,l)\).

**Q4.6**

Let \((\Omega)\) be the rectangle \(((0,a) \times (0,b))\). Find all eigenvalues and associated eigenfunctions of \(-\triangle u = \lambda u\) in \(\Omega\), \(u = 0\) on \(\partial \Omega\). *Hint:* Separation of variables.

**Q4.7**

Find a solution of Schrödinger's equation
$$i\hbar \psi_t = -\frac{\hbar^2}{2m} \triangle_x \psi + V(x) \psi \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}^1,$$
which satisfies the side condition
$$\int_{\mathbb{R}^n} |\psi(x,t)|^2 \, dx = 1,$$
provided \(|E| \in \mathbb{R}^n\) is an (eigenvalue) of the elliptic equation
$$\triangle u + \frac{2m}{\hbar^2} (E-V(x)) u = 0 \quad \text{in} \quad \mathbb{R}^n$$
under the side condition \(\int_{\mathbb{R}^n} |u|^2 \, dx = 1\), \(u: \mathbb{R}^n \mapsto \mathbb{C}\).

Here is
\[\psi: \mathbb{R}^n \times \mathbb{R}^1 \mapsto \mathbb{C}\]
Planck's constant \((\hbar)\) is a small positive constant and \(V(x)\) a given potential.

*Remark.* In the case of a hydrogen atom the potential is \(|V(x)| = -e/|x|\), \((e)\) is here a positive constant. Then eigenvalues are given by \(|E_n| = me^4/(2\hbar^2n^2)\), \(|n| \in \mathbb{N}\), see [22], pp. 202.

**Q4.8**

Find nonzero solutions by using separation of variables of \(u_{tt} = \triangle_x u\) in \(\Omega \times (0,\infty)\), \(u(x,t) = 0\) on \(\partial \Omega\), where \(\Omega\) is the circular cylinder \((x_1, x_2, x_3) \in \mathbb{R}^n: x_1^2 + x_2^2 < R^2, 0 < x_3 < h\).
Q4.9

Solve the initial value problem

\begin{eqnarray*}
3u_{tt}-4u_{xx} &=& 0 \\
u(x,0) &=& \sin x \\
u_t(x,0) &=& 1 . \\
\end{eqnarray*}

Q4.10

Solve the initial value problem

\begin{eqnarray*}
u_{tt}-c^2u_{xx} &=& x^2, t > 0, x \in \mathbb{R}^1 \\
u(x,0) &=& x \\
u_t(x,0) &=& 0 . \\
\end{eqnarray*}

\textit{Hint:} Find a solution of the differential equation independent on \(t\), and transform the above problem into an initial value problem with homogeneous differential equation by using this solution.

Q4.11

Find with the method of separation of variables nonzero solutions \(u(x,t)\), \(0 \leq x \leq 1, 0 \leq t < \infty\) of

\[ u_{tt}-u_{xx}+u=0 , \]

such that \(u(0,t)=0\), and \(u(1,t)=0\) for all \(t \in [0, \infty)\).

Q4.12

Find solutions of the equation

\[ u_{tt}-c^2u_{xx}=\lambda^2u, \lambda=\text{const.} \]

which can be written as

\[ u(x,t)=f(x^2-c^2t^2)=f(s), s:=x^2-c^2t^2 \]
with \( f(0) = K \), \( K \) a constant.

**Hint:** Transform equation for \( f(s) \) by using the substitution \( s = z^2/A \) with an appropriate constant \( A \) into Bessel's differential equation

\[
\frac{d^2}{dz^2} f(z) + \frac{1}{z} \frac{df}{dz} + \left( \frac{z^2}{A^2} - n^2 \right) f(z) = 0, \quad z > 0
\]

with \( n = 0 \).

**Remark.** The above differential equation for \( u \) is the transformed telegraph equation (see Section 4.4).

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**Q4.13**

Find the formula for the solution of the following Cauchy initial value problem \( u_{xy} = f(x, y) \), where \( S: y = ax + b \), \( a > 0 \), and the initial conditions on \( S \) are given by

\[
\begin{align*}
  u &= \alpha x + \beta y + \gamma, \\
  u_x &= \alpha, \\
  u_y &= \beta,
\end{align*}
\]

\( a, b, \alpha, \beta, \gamma \) constants.

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**Q4.14**

Find all eigenvalues \( \mu \) of

\[
\begin{align*}
  -q''(\theta) &= \mu q(\theta) \\
  q(\theta) &= q(\theta + 2\pi)
\end{align*}
\]

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**Contributors**

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