6.7: Inverse and Implicit Functions. Open and Closed Maps

I. "If \((f \in C D^1)\) at \((\vec{p};\cdot)\) then \((f)\) resembles a linear map (namely \((d f)\)) at \((\vec{p};\cdot)\)." Pursuing this basic idea, we first make precise our notion of "\((f \in C D^1)\) at \((\vec{p};\cdot)\)."

Definition 1

A map \((f : E' \rightarrow E)\) is continuously differentiable, or of class \((C D^1)\) (written \((f \in C D^1)\)) at \((\vec{p};\cdot)\) iff the following statement is true:

\[
\text{Given any } \varepsilon > 0, \text{ there is } \delta > 0 \text{ such that } f \text{ is differentiable on the globe } \overline{G} = \overline{G_{\vec{p}}(\delta)}, \text{ with } \|d f(\vec{x}; \cdot) - d f(\vec{p}; \cdot)\| < \varepsilon \text{ for all } \vec{x} \in \overline{G}.
\]

By Problem 10 in §5, this definition agrees with Definition 1 §5, but is no longer limited to the case \((E' = E^n)\) (written \((f \in C D^1)\)) at \((\vec{p};\cdot)\). See also Problems 1 and 2 below.

We now obtain the following result.

Theorem 1

Let \((E' = E^n)\) and \((E)\) be complete. If \((f : E' \rightarrow E)\) is of class \((C D^1)\) at \((\vec{p};\cdot)\) and if \((d f(\vec{x}; \cdot))\) is bijective (§6), then \((f)\) is one-to-one on some globe \((\overline{G} = \overline{G_{\vec{p}}(\delta)})\).

Thus \((f)\) "locally" resembles \((d f(\vec{p}; \cdot))\) in this respect.
Proof

Set \( \phi = d f(\vec{p} ; \cdot) \) and

\[
\|\phi^{-1}\| = \frac{1}{\varepsilon}
\]

(cf. Theorem 2 of §6).

By Definition 1, fix \( \delta > 0 \) so that for \( \vec{x} \in \overline{G} = \overline{G_{\vec{p}}(\delta)} \).

\[
\|d f(\vec{x} ; \cdot) - \phi\| < \frac{1}{2} \varepsilon.
\]

Then by Note 5 in §2,

\[
(\forall \vec{x} \in \overline{G}) (\forall \vec{u} \in E^\prime) \quad |d f(\vec{x} ; \vec{u}) - \phi(\vec{u})| \leq \frac{1}{2} \varepsilon |\vec{u}|.
\]

Now fix any \( \vec{r}, \vec{s} \in \overline{G}, \vec{r} \neq \vec{s}, \) and set \( \vec{u} = \vec{r} - \vec{s} \neq 0 \). Again, by Note 5 in §2,

\[
0 < \varepsilon |\vec{u}| \leq |\phi(\vec{u})|.
\]

By convexity, \( \overline{G} \supseteq I = L[\vec{s}, \vec{r}] \), so (1) holds for \( \vec{x} \in I, \vec{x} = \vec{s} + t \vec{u}, 0 \leq t \leq 1 \).

Noting this, set

\[
h(t) = f(\vec{s} + t \vec{u}) - t \phi(\vec{u}), \quad t \in E^1.
\]

Then for \( 0 \leq t \leq 1 \),

\[
\begin{aligned}
\begin{aligned}
|h'(t)| &= d f(\vec{s} + t \vec{u} ; \vec{u}) - \phi(\vec{u}) \\
&\leq \frac{\varepsilon}{2} |\vec{u}| \leq \frac{1}{2} |\phi(\vec{u})|.
\end{aligned}
\end{aligned}
\]

(Explain!) Now, by Corollary 1 in Chapter 5, §4,
As \( h(0) = f(\vec{s}) \) and
\[
|h(1) - f(\vec{s}) + \phi(\vec{u})| = f(\vec{r}) - \phi(\vec{u}),
\]
we obtain (even if \( \vec{r} = \vec{s} \))
\[
|f(\vec{r}) - f(\vec{s})| \leq \frac{1}{2} |\phi(\vec{u})| \quad (\vec{r}, \vec{s} \in \overline{G}, \vec{u} = \vec{r} - \vec{s}).
\]

But by the triangle law,
\[
|f(\vec{r}) - f(\vec{s}) - \phi(\vec{u})| \leq |f(\vec{r}) - f(\vec{s})| + |\phi(\vec{u})|.
\]

Thus
\[
|f(\vec{r}) - f(\vec{s})| \geq \frac{1}{2} |\phi(\vec{u})| \geq \frac{1}{2} \epsilon|\vec{u}| = \frac{1}{2} \epsilon|\vec{r} - \vec{s}| \quad \text{by (2)}.
\]

Hence \( f(\vec{r}) \neq f(\vec{s}) \) whenever \( \vec{r} \neq \vec{s} \) in \( \overline{G} \); so \( f \) is one-to-one on \( \overline{G} \), as claimed. \( \square \)

**Corollary \( \PageIndex{1} \) **

Under the assumptions of Theorem 1, the maps \( f \) and \( f^{-1} \) (the inverse of \( f \) restricted to \( \overline{G} \)) are uniformly continuous on \( \overline{G} \) and \( f(\overline{G}) \), respectively.

**Proof**

By (3),
\[
|f(\vec{r}) - f(\vec{s})| \leq |\phi(\vec{u})| + \frac{1}{2} |\phi(\vec{u})| \leq 2 |\phi(\vec{u})| \leq 2 \|\phi\| |\vec{u}| \quad (\vec{r}, \vec{s} \in \overline{G}).
\]

This implies uniform continuity for \( f \). (Why?)

Next, let \( g = f^{-1} \) on \( H = f(\overline{G}) \).

If \( \vec{x}, \vec{y} \in H \), let \( \vec{r} = g(\vec{x}) \) and \( \vec{s} = g(\vec{y}) \); so \( \vec{r}, \vec{s} \in \overline{G} \) with \( f(\vec{r}) = f(\vec{s}) \). Hence by (4),
\[
|\vec{x} - \vec{y}| \geq \frac{1}{2} |\epsilon| |g(\vec{x}) - g(\vec{y})|.
\]
proving all for \(g,\) too.\(\square\)

Again, \(f\) resembles \(\phi\) which is uniformly continuous, along with \(\phi^{-1}\).

II. We introduce the following definition.

Definition 2

A map \(f : (S, \rho) \rightarrow (T, \rho')\) is closed (open) on \(D \subseteq S\) iff, for any \(X \subseteq D\) the set \(f[X]\) is closed (open) in \(T\) whenever \(X\) is so in \(S\).

Note that continuous maps have such a property for inverse images (Problem 15 in Chapter 4, §2).

Corollary \(\PageIndex{2}\)

Under the assumptions of Theorem 1, \(f\) is closed on \(\overline{G}\) and so the set \(\overline{fG}\) is closed in \(E\).

Similarly for the map \(f^{-1}\) on \(\overline{fG}\).

Proof for \(E^\prime=E=E^n(C^n)\) (for the general case, see Problem 6)

Given any closed \((X \subset S, \rho) \rightarrow (T, \rho')\) we must show that \((f[X])\) is closed in \((E, \rho')\).

Now, as \((\overline{G}, \rho)\) is compact (Theorem 4 of Chapter 4, §6), and so is \((f[X])\) (Theorem 1 of Chapter 4, §8).

By Theorem 2 in Chapter 4, §6, \((f[X])\) is closed, as required.\(\square\)

For the rest of this section, we shall set \(E^\prime=E=E^n(C^n)\).

Theorem \(\PageIndex{2}\)

If \((E^\prime=C^n)\) in Theorem 1, with other assumptions unchanged, then \(f\) is open on the globe \(G=G_{\vec{p}}(\delta)\), with \(\delta\) sufficiently small.

Proof

We first prove the following lemma.

Lemma

\((f[G])\) contains a globe \(G_{\vec{q}}(\alpha)\) where \(\vec{q}=f(\vec{p})\).
Proof

Indeed, let

\[ \alpha = \frac{1}{4} \varepsilon \delta, \]

where \( \delta \) and \( \varepsilon \) are as in the proof of Theorem 1. (We continue the notation and formulas of that proof.) Fix any \( \vec{c} \in G_{\vec{q}}(\alpha) \); so

\[ |\vec{c} - \vec{q}| < \alpha = \frac{1}{4} \varepsilon \delta. \]

Set \( h = |f^{-1}(\vec{c})| \) on \( (E^{'prime}) \). As \( f \) is uniformly continuous on \( \overline{G} \), so is \( h \).

Now, \( \overline{G} \) is compact in \( (E^{n}) \); so Theorem 2(ii) in Chapter 4, §8, yields a point \( \vec{r} \in \overline{G} \) such that

\[ h(\vec{r}) = \min h(\overline{G}). \]

We claim that \( \vec{r} \) is in \( G \) (the interior of \( \overline{G} \)). Otherwise, \( |\vec{r} - \vec{p}| = \delta \); for by (4),

\[
\begin{aligned}
2 \alpha &= \frac{1}{2} \varepsilon \delta = \frac{1}{2} \varepsilon |\vec{r} - \vec{p}| \\
&\leq |f(\vec{r}) - f(\vec{p})| \\
&= |f(\vec{r}) - \vec{c}| + |\vec{c} - f(\vec{p})| \\
&= h(\vec{r}) + h(\vec{p}).
\end{aligned}
\]

But

\[ h(\vec{p}) = |\vec{c} - f(\vec{p})| = |\vec{c} - \vec{q}| < \alpha; \]

and so (7) yields

\[ h(\vec{p}) < \alpha < h(\vec{r}), \]

contrary to the minimality of \( h(\vec{r}) \) (see (6)). Thus \( |\vec{r} - \vec{p}| \) cannot equal \( |\delta| \).

We obtain \( |\vec{r} - \vec{p}| < \delta \) so \( \vec{r} \in G_{\vec{p}}(\delta) \) and \( f(G_{\vec{r}}) \in f(G) \). We shall now show that \( \vec{c} = f(\vec{r}) \).

To this end, we set \( \phi = f^{-1}(\vec{c}) \) and prove that \( \vec{v} \mapsto \phi(\vec{v}) \) is the identity on \( G \).

where

\[ \phi = d f(\vec{p}) ; \cdot \]
as before. Then
\[
\vec{v} = \phi(\vec{u}) = d f(\vec{p} ; \vec{u}).
\]
With \(\vec{r}\) as above, fix some
\[
\vec{s} = \vec{r} + t \vec{u} \quad (0 < t < 1)
\]
with \(t\) so small that \(\vec{s} \in G\) also. Then by formula (3),
\[
|f(\vec{s}) - f(\vec{r}) - \phi(t \vec{u})| \leq \frac{1}{2} |t \vec{v}|
\]
also,
\[
|h(\vec{s}) - h(\vec{r})| \leq |f(\vec{s}) - f(\vec{r}) - \phi(t \vec{u})| = (1-t)|\vec{v}| = (1-t) h(\vec{r})
\]
by our choice of \((\vec{v}, \vec{u})\) and \((h,.)\) Hence by the triangle law,
\[
|h(\vec{s}) - h(\vec{r})| \leq |f(\vec{s}) - f(\vec{r}) - \phi(t \vec{u})| + \phi(t \vec{u}) = |\phi(t \vec{u})| = |t \vec{v}|
\]
(Verify!)
As \((0 < t < 1,.)\) this implies \((h(\vec{r}) = 0)\) (otherwise, \((h(\vec{s}) < h(\vec{r}),.)\) violating (6)).

Thus, indeed,
\[
|f(\vec{v}) - f(\vec{r})| = 0.
\]
i.e.,
\[
\vec{c} = f(\vec{r}) \in f[G] \quad \text{for } \vec{r} \in G.
\]
But \(\vec{c}\) was an arbitrary point of \(G_{\vec{q}}(\alpha)\) Hence
\[
G_{\vec{q}}(\alpha) \subseteq f[G],
\]
proving the lemma.

Proof of Theorem 2. The lemma shows that \((f(\vec{p}))\) is in the interior of \((f[G])\) if \((\vec{p}, f, d f(\vec{p} ; \cdot),.)\) and \((\delta)\) are as in Theorem 1.

But Definition 1 implies that here \((f \in C D^1)\) on all of \((G)\) (see Problem 1).

Also, \((d f(\vec{x} ; \cdot))\) is bijective for any \((\vec{x} \in G)\) by our choice of \((G)\) and Theorems 1 and 2 in §6.

Thus \((f)\) maps all \((\vec{x} \in G)\) onto interior points of \((f[G])\); i.e., \((f)\) maps any open set \((X \subseteq G)\) onto an open
\( \langle f[X], \rangle \) as required.\(^{\text{\square}}\)

**Note 1.** A map

\( f : (S, \rho) \underset{\text{onto}}{\longleftrightarrow} (T, \rho^{\prime}) \)

is both open and closed ("clopen") iff \( f^{-1} \) is continuous - see Problem 15(iv)(v) in Chapter 4, §2, interchanging \( f \) and \( f^{-1} \).

Thus \( \phi = df(\vec{p} ; \cdot) \) in Theorem 1 is "clopen" on all of \( \langle E^{\prime}, \rho^{\prime}\rangle \).

Again, \( f \) locally resembles \( df(\vec{p} ; \cdot) \).

**III. The Inverse Function Theorem.** We now further pursue these ideas.

Theorem \( \PageIndex{3} \) (inverse functions)

Under the assumptions of Theorem 2, let \( g \) be the inverse of \( f[G] \) restricted to \( G = \{ \vec{p} \} \) \( \delta \).

Then \( g \in C D^{1} \) on \( f(G) \) and \( d g(\vec{y} ; \cdot) \) is the inverse of \( df(\vec{x} ; \cdot) \) whenever \( \vec{x} = g(\vec{y}) \).

Briefly: "The differential of the inverse is the inverse of the differential."

**Proof**

Fix any \( \vec{y} \in f(G) \) and \( \vec{x} = g(\vec{y}) \); so \( \vec{y} = f(\vec{x}) \) and \( \vec{x} \in G \) Let \( U = df(\vec{x} ; \cdot) \).

As noted above, \( U \) is bijective for every \( \vec{x} \in G \) by Theorems 1 and 2 in §6; so we may set \( V = U^{-1} \).

We must show that \( V = d g(\vec{y} ; \cdot) \).

To do this, give \( \vec{y} \) an arbitrary (variable) increment \( \Delta \vec{y} \) so small that \( \vec{y} + \Delta \vec{y} \) stays in \( f(G) \) (an open set by Theorem 2).

As \( g \) and \( f[G] \) are one-to-one, \( \Delta \vec{y} \) uniquely determines

\[ \Delta \vec{x} = g(\vec{y} + \Delta \vec{y}) - g(\vec{y}) = \vec{t}, \]

and vice versa:

\[ \Delta \vec{y} = f(\vec{x} + \vec{t}) - f(\vec{x}). \]

Here \( \Delta \vec{y} \) and \( \Delta \vec{t} \) are the mutually corresponding increments of \( \vec{y} = f(\vec{x}) \) and \( \vec{x} = g(\vec{y}) \). By continuity, \( \vec{y} \rightarrow 0 \) iff \( \vec{t} \rightarrow 0 \).
As \( U = df(\vec{x} ; \cdot) \),

\[
\lim _{\vec{t} \to \overrightarrow{0}} \frac{1}{|\vec{t}|}|f(\vec{x}+\vec{t})-f(\vec{t})-U(\vec{t})|=0,\]

or

\[
\lim _{\vec{t} \to \overrightarrow{0}} \frac{1}{|\vec{t}|}|F(\vec{t})|=0,\]

where

\[
F(\vec{t})=f(\vec{x}+\vec{t})-f(\vec{t})-U(\vec{t}).\]

As \( V = U^{-1} \), we have

\[
V(U(\vec{t}))=\vec{t}=g(\vec{y}+\Delta \vec{y})-g(\vec{y}).\]

So from (9),

\[
\begin{aligned}
V(F(\vec{t})) &= V(\Delta \vec{y})-\vec{t} \\
&= V(\Delta \vec{y})-\left[g(\vec{y}+\Delta \vec{y})-g(\vec{y})\right];
\end{aligned}
\]

that is,

\[
\frac{1}{|\Delta \vec{y}|}|g(\vec{y}+\Delta \vec{y})-g(\vec{y})-V(\Delta \vec{y})|=rac{|V(F(\vec{t}))|}{|\Delta \vec{y}|}, \quad \Delta \vec{y} \neq \overrightarrow{0}.\]

Now, formula (4), with \( \vec{r} = \vec{x}, \vec{s} = \vec{x}+\vec{t}, \) and \( \vec{u} = \vec{t} \), shows that

\[
|f(\vec{x}+\vec{t})-f(\vec{x})| \geq \frac{1}{2} \varepsilon |\vec{t}|;\]

i.e., \( |\Delta \vec{y}| \geq \frac{1}{2} \varepsilon |\vec{t}| \). Hence by (8),

\[
\begin{aligned}
\frac{|V(F(\vec{t}))|}{|\Delta \vec{y}|} &\leq \frac{|V(F(\vec{t}))|}{\frac{1}{2} \varepsilon |\vec{t}|} = \frac{2}{\varepsilon} \|V\| \frac{1}{|\vec{t}|}|F(\vec{t})| \to 0 \text{ as } \vec{t} \to \overrightarrow{0}.
\end{aligned}
\]

Since \( \vec{t} \to \overrightarrow{0} \) as \( \Delta \vec{y} \to \overrightarrow{0} \) (change of variables!), the expression (10) tends to 0 as \( \Delta \vec{y} \to \overrightarrow{0} \).

By definition, then, \( g() \) is differentiable at \( \vec{y} \) with \( \langle d g(\vec{y}); = V = U^{-1} \rangle \).

Moreover, Corollary 3 in §6, applies here. Thus
\[
\left(\forall \delta' > 0\right) \left(\exists \delta'' > 0\right) \quad \|U-W\| < \delta'' \Rightarrow \|U^{-1}-W^{-1}\| < \delta'.
\]

Taking here \(U^{-1}=d g(\vec{y})\) and \(W^{-1}=d g(\vec{y}+\Delta \vec{y})\), we see that \(g \in C D^{1}\) near \(\langle \vec{y}, \cdot \rangle\). This completes the proof. \(\square\)

**Note 2.** If \(E^\prime=E=E^n(C^n)\), the bijectivity of \(\phi=d f(\vec{p} ; \cdot)\) is equivalent to

\[
\det[\phi]=\det[f^\prime(\vec{p})] \neq 0
\]

(Theorem 1 of §6).

In this case, the fact that \(f\) is one-to-one on \(G=G_{\vec{p}}(\delta)\) means, componentwise (see Note 3 in §6), that the system of \((n)\) equations

\[
[f_{\cdot i}](\vec{x})=f(\vec{x}_{1}, \ldots, \vec{x}_{n})=y_{i}, \quad i=1, \ldots, n,
\]

has a unique solution for the \((n)\) unknowns \(x_{\cdot k}\) as long as

\[
\left(\begin{array}{l}
\left(\forall \left(\begin{array}{c}
\vec{x}_{1}, \ldots, \vec{x}_{n}
\end{array}\right) \Rightarrow y_{\cdot i}, \quad \forall i=1, \ldots, n.
\end{array}\right)
\right)
\]

Theorem 3 shows that this solution has the form

\[
(x_{\cdot k}=g_{\cdot k}(\vec{y}), \quad \forall k=1, \ldots, n,
\]

where the \(g_{\cdot k}\) are of class \(C D^{1}\) on \(f[G]\) provided the \(f_{\cdot i}\) are of class \(C D^{1}\) near \(\vec{y}\) and \(\det[f_{\cdot i}](\vec{p}) \neq 0\). Here

\[
\det[f_{\cdot i}](\vec{p})=J_{f}(\vec{p}),
\]

as in §6.

Thus again \(f\) "locally" resembles a linear map, \(\phi=d f(\vec{p} ; \cdot)\).

**IV. The Implicit Function Theorem.** Generalizing, we now ask, what about solving \((n)\) equations in \((n+m)\) unknowns

\[
(\begin{array}{l}
x_{\cdot 1}, \ldots, x_{\cdot n}, y_{\cdot 1}, \ldots, y_{\cdot m}
\end{array})\]  
Say, we want to solve

\[
[f_{\cdot k}](\vec{x}_{1}, \ldots, \vec{x}_{n}, \vec{y}_{1}, \ldots, \vec{y}_{m})=0, \quad \forall k=1,2, \ldots, n,
\]

for the first \((n)\) unknowns (or variables) \(x_{\cdot k}\) thus expressing them as

\[
(x_{\cdot k}=H_{\cdot k}(\vec{y}_{1}, \ldots, \vec{y}_{m}), \quad \forall k=1, \ldots, n)
\]

with \(H_{\cdot k} : E^m \rightarrow E^n\).
Let us set \( \vec{x} = (x_1, \ldots, x_n), \vec{y} = (y_1, \ldots, y_m) \) and
\[ (\vec{x}, \vec{y}) = (x_1, \ldots, x_n, y_1, \ldots, y_m) \]
so that \( (\vec{x}, \vec{y}) \in E^{n+m}(\mathbb{C}^{n+m}) \).

Thus the system of equations (11) simplifies to
\[ f_k(\vec{x}, \vec{y}) = 0, \quad k = 1, \ldots, n \]
or
\[ f(\vec{x}, \vec{y}) = \overrightarrow{0}, \]
where \( f = (f_1, \ldots, f_n) \) is a map of \( E^{n+m}(\mathbb{C}^{n+m}) \) into \( E^n(\mathbb{C}^n) \); \( f \) is a function of \( n+m \) variables, but it has \( n \) components \( (f_k) \), i.e.,
\[ f(\vec{x}, \vec{y}) = f(x_1, \ldots, x_n, y_1, \ldots, y_m) \]
is a vector in \( E^n(\mathbb{C}^n) \).

Theorem \( \PageIndex{4} \) (implicit functions)

Let \( E' = E^{n+m}(\mathbb{C}^{n+m}), E = E^n(\mathbb{C}^n) \), and let \( f : E' \to E \) be of class \( C^1 \) near
\[ (\vec{p}, \vec{q}) = (p_1, \ldots, p_n, q_1, \ldots, q_m), \quad \vec{p} \in E^n(\mathbb{C}^n), \vec{q} \in E^m(\mathbb{C}^m). \]
Let \( \phi \) be the \( n \times n \) matrix
\[ (D_j f_k(\vec{p}, \vec{q})), \quad j, k = 1, \ldots, n. \]
If \( \det(\phi) \neq 0 \) and if \( f(\vec{p}, \vec{q}) = \overrightarrow{0} \) then there are open sets
\[ P \subseteq E^n(\mathbb{C}^n), Q \subseteq E^m(\mathbb{C}^m), \]
with \( \vec{p} \in P \) and \( \vec{q} \in Q \) for which there is a unique map
\[ H : Q \to P \]
with
\[ f(H(\vec{y}), \vec{y}) = \overrightarrow{0} \]
for all \( \vec{y} \in Q \); furthermore, \( H \) is of class \( C^1 \) on \( Q \).
Thus \((\text{vec}\{x\}) = H(\text{vec}\{y\})\) is a solution of (11) in vector form.

**Proof**

With the above notation, set

\[
F(\text{vec}\{x\}, \text{vec}\{y\}) = (f(\text{vec}\{x\}, \text{vec}\{y\}), \text{vec}\{y\}), \quad F : E^{\prime} \rightarrow E^{\prime}.
\]

Then

\[
F(\text{vec}\{p\}, \text{vec}\{q\}) = (f(\text{vec}\{p\}, \text{vec}\{q\}), \text{vec}\{q\}) = (\overrightarrow{0}, \text{vec}\{q\}),
\]

since \((f(\text{vec}\{p\}, \text{vec}\{q\}) = \overrightarrow{0})\).

As \((f \in C D^{1})\) near \(((\text{vec}\{p\}, \text{vec}\{q\}))\), so is \((F)\) (verify componentwise via Problem 9(ii) in §3 and Definition 1 of §5).

By Theorem 4, §3, \((\text{det} \left[F^{\prime}(\text{vec}\{p\}, \text{vec}\{q\})\right] \neq 0)\) (explain!).

Thus Theorem 1 above shows that \((F)\) is one-to-one on some globe \((G)\) about \(((\text{vec}\{p\}, \text{vec}\{q\})).\)

Clearly \((G)\) contains an open interval about \(((\text{vec}\{p\}, \text{vec}\{q\})).\) We denote it by \((P \times Q)\) where \((\text{vec}\{p\} \in P, \text{vec}\{q\} \in Q; P)\) is open in \((E^{n}\left[\leftarrow \text{left}(C^{m}\left[\leftarrow \text{right}))\right)\right.\) and \((Q)\) is open in \((E^{m}\left[\leftarrow \text{left}(C^{m}\left[\leftarrow \text{right}))\right)\right.\)

By Theorem 3, \((F \mid P \times Q)\) \((\text{F})\) restricted to \((P \times Q)\) has an inverse

\[
g : A \underset{\text{onto}}{\longleftrightarrow} P \times Q,
\]

where \((A = F(P \times Q))\) is open in \((E^{\prime}\left[\leftarrow \text{prime})\right.\) (Theorem 2), and \((g \in C D^{1})\) on \((A).\) Let the map

\[
g(\text{vec}\{x\}, \text{vec}\{y\}) = (u(\text{vec}\{x\}, \text{vec}\{y\}), \text{vec}\{y\})
\]

exactly as \((F(\text{vec}\{x\}, \text{vec}\{y\}) = (f(\text{vec}\{x\}, \text{vec}\{y\}), \text{vec}\{y\})).\) Also, \((u : A \rightarrow P)\) is of class \((C D^{1})\) on \((A)\) as \((g)\) is (explain!).

Now set

\[
H(\text{vec}\{y\}) = u(\overrightarrow{0}, \text{vec}\{y\});
\]

here \((\text{vec}\{y\}) \in Q)\) while
$\overrightarrow{v}, \vec{y}) \in A=F[P \times Q]$.

for $F$ preserves $(\vec{y})$ (the last $m'$ coordinates). Also set

$\alpha(\vec{x}, \vec{y})=\vec{x}.$

Then $f=\alpha \circ F$ (why?), and

$f(\overrightarrow{0}, \vec{y})=f(u(\overrightarrow{0}, \vec{y}), \vec{y})=f(g(\overrightarrow{0}, \vec{y}))=\alpha(F(g(\overrightarrow{0}, \vec{y})))=\alpha(\overrightarrow{0}, \vec{y})=\overrightarrow{0}$

by our choice of $(\alpha)$ and $(g)$ (inverse to $F$). Thus

$f(\overrightarrow{0}, \vec{y})=\overrightarrow{0}, \quad \vec{y} \in Q,$

as desired.

Moreover, as $H(\vec{y})=u(\overrightarrow{0}, \vec{y})$, we have

$\frac{\partial}{\partial y_i} H(\vec{y})=\frac{\partial}{\partial y_i} u(\overrightarrow{0}, \vec{y}), \quad \vec{y} \in Q, \ i \leq m.$

As $u \in C_D^1$ all $\frac{\partial u}{\partial y_i}$ are continuous (Definition 1 in §5); hence so are the $\frac{\partial H}{\partial y_i}$. Thus by Theorem 3 in §3, $H \in C_D^1$ on $Q$.

Finally, $H$ is unique for the given $(P, Q)$ for

$f(\vec{x}, \vec{y})=\overrightarrow{0} \quad \Rightarrow \quad f(\vec{x}, \vec{y}), \vec{y})=g(\overrightarrow{0}, \vec{y}) \Rightarrow \quad \vec{x}=u(\overrightarrow{0}, \vec{y})=H(\vec{y}).$

Thus $f(\vec{x}, \vec{y})=\overrightarrow{0}$ implies $\vec{x}=H(\vec{y})$ so $H(\vec{y})$ is the only solution for $f(\vec{x}, \vec{y})=\overrightarrow{0}$.

**Note 3.** $H$ is said to be implicitly defined by the equation $f(\vec{x}, \vec{y})=\overrightarrow{0}$. In this sense we say that $H(\vec{y})$ is an implicit function, given by $f(\vec{x}, \vec{y})=\overrightarrow{0}$.

Similarly, under suitable assumptions, $f(\vec{x}, \vec{y})=\overrightarrow{0}$ defines $\vec{y}$ as a function of $\vec{x}$.

**Note 4.** While $H$ is unique for a given neighborhood $(P \times Q)$ of $((\vec{p}, \vec{q}))$, another implicit function may result if $(P \times Q)$ or $((\vec{p}, \vec{q}))$ is changed.
For example, let
\[ f(x, y) = x^2 + y^2 - 25 \]
(a polynomial; hence \( f \in C D^1 \) on all of \( E^2 \)). Geometrically, \( x^2 + y^2 - 25 = 0 \) describes a circle.

Solving for \( x \), we get \( x = \pm \sqrt{25 - y^2} \). Thus we have two functions:
\[ H_1(y) = +\sqrt{25 - y^2} \]
and
\[ H_2(y) = -\sqrt{25 - y^2} \]
If \( (P \times Q) \) is in the upper part of the circle, the resulting function is \( H_1 \). Otherwise, it is \( H_2 \). See Figure 28.

\[ \text{Figure 28} \]

V. Implicit Differentiation. Theorem 4 only states the existence (and uniqueness) of a solution, but does not show how to find it, in general.

The knowledge itself that \( H \in C D^1 \) exists, however, enables us to use its derivative or partials and compute it by implicit differentiation, known from calculus.

Examples

(a) Let \( f(x, y) = x^2 + y^2 - 25 = 0 \) as above.

This time treating \( y \) as an implicit function of \( x, y = H(x) \) and writing \( y' \) for \( H'(x) \) we differentiate both sides of \( x^2 + y^2 - 25 = 0 \) with respect to \( x \) using the chain rule for the term \( y^2 = [H(x)]^2 \).
This yields \((2x + 2yy' = 0,\) whence \((y' = -x / y).\)

Actually (see Note 4), two functions are involved: \((y = \pm \sqrt{25 - x^2};\) but both satisfy \((x^2 + y^2 - 25 = 0;\) so the result \((y' = -x / y)\) applies to both.

Of course, this method is possible only if the derivative \(y'\) is known to exist. This is why Theorem 4 is important.

(b) Let

\[
\begin{align*}
f(x, y, z) &= x^2 + y^2 + z^2 - 1 = 0, \quad x, y, z \in \mathbb{E}^1. \\
\end{align*}
\]

Again \(f\) satisfies Theorem 4 for suitable \((x, y, z)\).

Setting \((z = H(x, y));\) differentiate the equation \((f(x, y, z) = 0);\) partially with respect to \((x);\) and \((y);\) From the resulting two equations, obtain \((\frac{\partial z}{\partial x})\) and \((\frac{\partial z}{\partial y});\)