2.2.1: A Linearization Method

We can transform the inhomogeneous Equation (2.2.1) into a homogeneous linear equation for an unknown function of three variables by the following trick.

We are looking for a function \( \psi(x,y,u) \) such that the solution \( u=u(x,y) \) of Equation (2.2.1) is defined implicitly by \( \psi(x,y,u)=\text{const.} \). Assume there is such a function \( \psi \) and let \( u \) be a solution of (2.2.1), then

\[
\psi_x + \psi_u u_x = 0, \quad \psi_y + \psi_u u_y = 0.
\]

Assume \( \psi_u \neq 0 \), then

\[
\begin{align*}
u_x &= -\frac{\psi_x}{\psi_u}, \\
u_y &= -\frac{\psi_y}{\psi_u}.
\end{align*}
\]

From (2.2.1) we obtain

\[
\begin{equation}
\begin{aligned}
a_1(x,y,z)\psi_x + a_2(x,y,z)\psi_y + a_3(x,y,z)\psi_z &= 0,
\end{aligned}
\end{equation}
\]

where \( z:=u \).

We consider the associated system of characteristic equations

\[
\begin{align*}
x'(t) &= a_1(x,y,z), \\
&= a_2(x,y,z) \\
&= a_3(x,y,z)
\end{align*}
\]
\[\begin{align*}
y'(t) &= a_2(x,y,z) \\
z'(t) &= a_3(x,y,z).
\end{align*}\]

One arrives at this system by the same arguments as in the two-dimensional case above.

**Proposition 2.2.** (i) Assume \(w \in C^1\), \(w=w(x,y,z)\), is an integral, i. e., it is constant along each fixed solution of (\ref{homthree}), then \(\psi=w(x,y,z)\) is a solution of (\ref{homthree}).

(ii) The function \(z=u(x,y)\), implicitly defined through \(\psi(x,u,z) = \text{const.}\), is a solution of (\ref{2.2.1}), provided that \(\psi_z \neq 0\).

(iii) Let \(z=u(x,y)\) be a solution of (\ref{2.2.1}) and let \((x(t),y(t))\) be a solution of
\[\begin{align*}
x'(t) &= a_1(x,y,u(x,y)), \\
y'(t) &= a_2(x,y,u(x,y)),
\end{align*}\]
then \(z(t)=u(x(t),y(t))\) satisfies the third of the above characteristic equations.

**Proof.** Exercise.

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**Contributors**

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