2.2.2: Initial Value Problem of Cauchy

Consider again the quasilinear equation

\[(a_1(x,y,u)u_x + a_2(x,y,u)u_y = a_3(x,y,u)).\]

Let

\[\Gamma: \quad x = x_0(s), \quad y = y_0(s), \quad z = z_0(s), \quad s_1 \leq s \leq s_2, \quad -\infty < s_1 < s_2 < +\infty\]

be a regular curve in \(\mathbb{R}^3\) and denote by \(\mathcal{C}\) the orthogonal projection of \(\Gamma\) onto the \((x,y)\)-plane, i.e.,

\[\mathcal{C}: \quad x = x_0(s), \quad y = y_0(s).\]

Initial value problem of Cauchy: Find a \(C^1\)-solution \(u = u(x,y)\) of \((\star)\) such that \(u(x_0(s), y_0(s)) = z_0(s)\), i.e., we seek a surface \(\mathcal{S}\) defined by \(z = u(x,y)\) which contains the curve \(\Gamma\).
Figure 2.2.2.1: Cauchy initial value problem

**Definition.** The curve \( \Gamma \) is said to be non-characteristic if

\[
x_0'(s)a_2(x_0(s),y_0(s))-y_0'(s)a_1(x_0(s),y_0(s))\neq 0.
\]

**Theorem 2.1.** Assume \( a_1, a_2, a_3 \in C^1 \) in their arguments, the initial data \( x_0, y_0, z_0 \in C^1[s_1,s_2] \) and \( \Gamma \) is non-characteristic.

Then there is a neighborhood of \( \cal{C} \) such that there exists exactly one solution \( \cal{u} \) of the Cauchy initial value problem.

**Proof.** (i) Existence. Consider the following initial value problem for the system of characteristic equations to \((\star)\):

\[
\begin{align*}
x'(t) &= a_1(x,y,z) \\
y'(t) &= a_2(x,y,z) \\
z'(t) &= a_3(x,y,z)
\end{align*}
\]

with the initial conditions

\[
\begin{align*}
x(s,0) &= x_0(s) \\
y(s,0) &= y_0(s) \\
z(s,0) &= z_0(s).
\end{align*}
\]

Let \( (x=x(s,t)), (y=y(s,t)), (z=z(s,t)) \) be the solution, \( (s_1 \leq s \leq s_2), (|t|<\eta) \) for an \( \eta>0 \). We will show that this set of curves, see Figure 2.2.2.1, defines a surface. To show this, we consider the inverse functions \( (s=s(x,y)), (t=t(x,y)) \) of \( (x=x(s,t)), (y=y(s,t)) \) and show that \( (z(s(x,y),t(x,y))) \) is a solution of the initial problem of Cauchy. The inverse functions \( (s) \) and \( (t) \) exist in a neighborhood of \( (t=0) \) since
\[
\det \frac{\partial(x,y)}{\partial(s,t)}\Big|_{t=0} = \left| \begin{array}{cc} x_s & x_t \\ y_s & y_t \end{array} \right|_{t=0} = x'_0(s)a_2 - y'_0(s)a_1 \neq 0,
\]
and the initial curve \(\Gamma\) is non-characteristic by assumption.

Set
\[
u(x,y) := z(s(x,y),t(x,y)),
\]
then \(\{u\}\) satisfies the initial condition since
\[
u(x,y)|_{t=0} = z(s,0) = z_0(s).
\]
The following calculation shows that \(\{u\}\) is also a solution of the differential equation (\(\star\)).
\[
\begin{align*}
a_1 u_x + a_2 u_y &= a_1(z_{ss} x + z_{tt} x) + a_2(z_{ss} y + z_{tt} y) \\
&= a_1(s_x z_{ss} + s_y z_{tt}) + a_2(t_x z_{ss} + t_y z_{tt}) \\
&= a_3
\end{align*}
\]
since \(0 = s_t = s_xx_t + s_yy_t\) and \(1 = t_t = t_xx_t + t_yy_t\).

(ii) Uniqueness. Suppose that \(\{v(x,y)\}\) is a second solution. Consider a point \(\{(x',y')\}\) in a neighborhood of the curve \(\{(x_0(s),y(s))\}, \{(s_1 - \epsilon \leq s \leq s_2 + \epsilon)\}\) small. The inverse parameters are \(\{s' = (x',y')\}, \{t' = (t',y')\}\), see Figure 2.2.2.2.
be the solution of the above initial value problem for the characteristic differential equations with the initial data

\[ x(s',0) = x_0(s'), \quad y(s',0) = y_0(s'), \quad z(s',0) = z_0(s'). \]

According to its construction this curve is on the surface \(\mathcal{S}\) defined by \(u=u(x,y)\) and \(u(x',y')=z(s',t')\). Set

\[ \psi(t) := v(x(t),y(t)) - z(t), \]

then

\[
\begin{align*}
\psi'(t) &= v_{xx}' + v_{yy}' - z' \\
&= x_{xx}a_1 + v_{ya}' - a_3 = 0
\end{align*}
\]

and

\[ \psi(0) = v(x(s',0),y(s',0)) - z(s',0) = 0 \]

since \(v\) is a solution of the differential equation and satisfies the initial condition by assumption. Thus, \(\psi(t)\equiv0\), i.e.,

\[ v(x(s',t),y(s',t)) - z(s',t) = 0. \]

Set \(t=t'\), then

\[ v(x',y') - z(s',t') = 0, \]

which shows that \(v(x',y')=u(x',y')\) because of \(z(s',t')=u(x',y')\).

Remark. In general, there is no uniqueness if the initial curve \(\Gamma\) is a characteristic curve, see an exercise and Figure 2.2.2.3, which illustrates this case.
Examples

Example 2.2.2.1:

Consider the Cauchy initial value problem
\[ u_x + u_y = 0 \]
with the initial data
\[ x_0(s) = s, \quad y_0(s) = 1, \quad z_0(s) \text{ is a given } C^1 \text{-function}. \]
These initial data are non-characteristic since \((y_0' a_1 - x_0' a_2 = -1)\). The solution of the associated system of characteristic equations
\[ x'(t) = 1, \quad y'(t) = 1, \quad u'(t) = 0 \]
with the initial conditions
\[ x(s, 0) = x_0(s), \quad y(s, 0) = y_0(s), \quad z(s, 0) = z_0(s) \]
is given by
\[ x = t + x_0(s), \quad y = t + y_0(s), \quad z = z_0(s), \]
i. e.,
\[ x = t + s, \quad y = t + 1, \quad z = z_0(s). \]

It follows \( s = x - y + 1, \quad t = y - 1 \) and that \( u = z_0(x - y + 1) \) is the solution of the Cauchy initial value problem.

Example 2.2.2.2:

A problem from kinetics in chemistry. Consider for \( x \geq 0, \quad y \geq 0 \) the problem

\[ u_x + u_y = \left( k_0e^{-k_1x} + k_2 \right)(1 - u) \]

with initial data

\[ u(x,0) = 0, \quad x > 0, \quad \mbox{and} \quad u(0,y) = u_0(y), \quad y > 0. \]

Here the constants \( k_j \) are positive, these constants define the velocity of the reactions in consideration, and the function \( u_0(y) \) is given. The variable \( x \) is the time and \( y \) is the height of a tube, for example, in which the chemical reaction takes place, and \( u \) is the concentration of the chemical substance.

In contrast to our previous assumptions, the initial data are not in \( C^1 \). The projection \( \mathcal{C}_1 \cup \mathcal{C}_2 \) of the initial curve onto the \( (x,y) \)-plane has a corner at the origin, see Figure 2.2.2.4.

The associated system of characteristic equations is

\[ x'(t) = 1, \quad y'(t) = 1, \quad z'(t) = \left( k_0e^{-k_1x} + k_2 \right)(1 - z). \]

It follows \( (x = t + c_1), \quad (y = t + c_2) \) with constants \( c_j \). Thus the projection of the characteristic curves on the \( ((x,y)) \)-plane are straight lines parallel to \( y = x \). We will solve the initial value problems in the domains \( \Omega_1 \) and \( \Omega_2 \), see Figure 2.2.2.4, separately.
(i) The initial value problem in $\Omega_1$. The initial data are
$$\begin{align*}
x_0(s) &= s, \\
y_0(s) &= 0, \\
z_0(0) &= 0, \\
\end{align*}$$
where $s \geq 0$.

It follows
$$x(s,t) = t + s, \\
y(s,t) = t.$$ 
Thus
$$z'(t) = (k_0 e^{-k_1(t+s)} + k_2)(1-z), \\
z(0) = 0.$$ 

The solution of this initial value problem is given by
$$z(s,t) = 1 - e^{-k_1(s+t)} + k_2.$$ 

Consequently
$$u_1(x,y) = 1 - e^{-k_1x} + k_2.$$ 

(ii) The initial value problem in $\Omega_2$. The initial data are here
$$\begin{align*}
x_0(s) &= 0, \\
y_0(s) &= s, \\
z_0(0) &= u_0(s), \\
\end{align*}$$
where $s \geq 0$.

It follows
$$x(s,t) = t, \\
y(s,t) = t + s.$$ 
Thus
$$z'(t) = (k_0 e^{-k_1t} + k_2)(1-z), \\
z(0) = 0.$$ 

The solution of this initial value problem is given by
$$z(s,t) = 1 - (1-u_0(s)) \exp\left(\frac{k_0}{k_1} e^{-k_1t} - k_2 t - \frac{k_0}{k_1}\right).$$
Consequently
\[ u_2(x,y)=1-(1-u_0(y-x))\exp\left(\frac{k_0}{k_1}e^{-k_1x}-k_2x\right) \]
is the solution in \(\Omega_2\).

If \(x=y\), then
\[
\begin{align*}
u_1(x,y) &= 1 - \exp\left(\frac{k_0}{k_1}e^{-k_1x}-k_2x\right) \\
u_2(x,y) &= 1 - (1-u_0(0))\exp\left(\frac{k_0}{k_1}e^{-k_1x}-k_2x\right).
\end{align*}
\]
If \(u_0(0)>0\), then \(u_1<u_2\) if \(x=y\), i.e., there is a jump of the concentration of the substrate along its burning front defined by \(x=y\).

**Remark.** Such a problem with discontinuous initial data is called *Riemann problem*. See an exercise for another Riemann problem.

The case that a solution of the equation is known

Here we will see that we get immediately a solution of the Cauchy initial value problem if a solution of the *homogeneous linear equation*
\[
a_1(x,y)u_x+a_2(x,y)u_y=0
\]
is known.

Let
\[
x_0(s),\ y_0(s),\ z_0(s),\ s_1<s<s_2
\]
be the initial data and let \(u=\phi(x,y)\) be a solution of the differential equation. We assume that
\[
\phi_x(x_0(s),y_0(s))x_0'(s)+\phi_y(x_0(s),y_0(s))y_0'(s)\neq0
\]
is satisfied. Set
\[
g(s)=\phi(x_0(s),y_0(s))
\]
and let \(s=h(g)\) be the inverse function.

*The solution of the Cauchy initial problem is given by \(u_0(h(\phi(x,y)))\right)\).*
This follows since in the problem considered a composition of a solution is a solution again, see an exercise, and since

$$u_0\left(h(\phi(x_0(s),y_0(s))\right)=u_0(h(g))=u_0(s).$$

Example 2.2.2.3:

Consider equation

$$u_x+u_y=0$$

with initial data

$$x_0(s)=s, \ y_0(s)=1, \ u_0(s) \text{ is a given function}.$$  

A solution of the differential equation is $\phi(x,y)=x-y$. Thus

$$\phi(x_0(s),y_0(s))=s-1$$

and

$$u_0(\phi+1)=u_0(x-y+1)$$

is the solution of the problem.

Contributors

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