7.3.1: Boundary Value Problems: Dirichlet Problem

The Dirichlet problem (first boundary value problem) is to find a solution \(u \in C^2(\Omega) \cap C(\overline{\Omega})\) of
\[
\begin{align}
\triangle u &= 0 \quad \text{in} \; \Omega \tag{7.3.1.1} \\
u &= \Phi \quad \text{on} \; \partial \Omega, \tag{7.3.1.2}
\end{align}
\]
where \(\Phi\) is given and continuous on \(\partial \Omega\).

**Proposition 7.4.** Assume \((\Omega)\) is bounded, then a solution to the Dirichlet problem is uniquely determined.

**Proof.** Maximum principle.

**Remark.** The previous result fails if we take away in the boundary condition (ref{D2}) one point from the boundary as the following example shows. Let \((\Omega)\subset \mathbb{R}^2\) be the domain
\[
\Omega = \{x \in B_1(0) : x_2 > 0\},
\]

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Assume \( u \in C^2(\Omega) \cap C(\overline{\Omega}\setminus\{0\}) \) is a solution of
\[
\begin{align*}
\Delta u &= 0 \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial \Omega \setminus \{0\}.
\end{align*}
\]
This problem has solutions \( u \equiv 0 \) and \( u = \text{Im}(z + z^{-1}) \), where \( z = x_1 + ix_2 \). Concerning another example see an exercise.

In contrast to this behavior of the Laplace equation, one has uniqueness if \( \Delta u = 0 \) is replaced by the minimal surface equation
\[
\frac{\partial}{\partial x_1} \left( \frac{u_{x_1}}{\sqrt{1 + |\nabla u|^2}} \right) + \frac{\partial}{\partial x_2} \left( \frac{u_{x_2}}{\sqrt{1 + |\nabla u|^2}} \right) = 0.
\]

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