8.9: Riemann Integration. Stieltjes Integrals

I. In this section, \(\mathcal{C}\) is the family of all intervals in \(E^n\) and \(m\) is an additive finite premeasure on \(\mathcal{C}\) (or \(\mathcal{C}_s\)), such as the volume function \(v\) (Chapter 7, §§1-2).

By a \(\mathcal{C}\)-partition of \(A \in \mathcal{C}\) (or \(A \in \mathcal{C}_s\)), we mean a finite family

\[\mathcal{P} = \{A_i\} \subset \mathcal{C}\]

such that

\[A = \bigcup_{i} A_i \text{ (disjoint)}\]

As we noted in §5, the Riemann integral,

\[R \int_A f = R \int_A f dm\]

of \(f : E^n \to E^1\) can be defined as its Lebesgue counterpart,

\[\int_A f\]

with elementary maps replaced by simple step functions ("\(\mathcal{C}\)-simple" maps.) Equivalently, one can use the following construction, due to J. G. Darboux.

Definitions

(a) Given \(f : E^n \to E^*\) and a \(\mathcal{C}\)-partition
\[ \mathcal{P} = \{ A_1, \ldots, A_q \} \]

of \( \{A,\} \) we define the lower and upper Darboux sums, \( \underline{S}(f, \mathcal{P}) \) and \( \overline{S}(f, \mathcal{P}) \) of \( f \) over \( \mathcal{P} \) (with respect to \( \{m\} \)) by

\[
\underline{S}(f, \mathcal{P}) = \sum_{i=1}^{q} m A_i \cdot \inf f[A_i] \text{ and } \overline{S}(f, \mathcal{P}) = \sum_{i=1}^{q} m A_i \cdot \sup f[A_i] .
\]

(b) The lower and upper Riemann integrals ("R-integrals") of \( f \) on \( \{A\} \) (with respect to \( \{m\} \)) are

\[
\Runderline{\int}_{A} f = \Runderline{\int}_{A} f dm = \sup_{\mathcal{P}} \underline{S}(f, \mathcal{P}) \text{ and } \Roverline{\int}_{A} f = \Roverline{\int}_{A} f dm = \inf_{\mathcal{P}} \overline{S}(f, \mathcal{P}),
\]

where the "inf" and "sup" are taken over all \( \{\mathcal{C}\} \)-partitions \( \mathcal{P} \) of \( \{A\} \).

(c) We say that \( f \) is Riemann-integrable ("R-integrable") with respect to \( \{m\} \) on \( \{A\} \) iff \( f \) is bounded on \( \{A\} \) and

\[
\Runderline{\int}_{A} f = \Roverline{\int}_{A} f \]

We then set

\[
\Rint_{A} f = \Runderline{\int}_{A} f = \Roverline{\int}_{A} f dm = \Rint_{A} f \ dm
\]

and call it the Riemann integral ("R-integral") of \( f \) on \( \{A\} \) "Classical" notation:

\[
\Rint_{A} f(x) dm(x)
\]

If \( \{A=[a, b] \subset E^{1}\} \) we also write

\[
\Rint_{a}^{b} f(x) \ dm(x) \]

instead.

If \( \{m\} \) is Lebesgue measure (or premeasure) in \( \{E^{1}\} \) we write "\( (dx)" for "\( (dm(x)) \)."

For Lebesgue integrals, we replace "\( (R)\)" by "\( (L)\)," or we simply omit "\( (R,\)".

If \( f \) is R-integrable on \( \{A,\} \) we also say that

\[
\Rint_{A} f \]

exists (note that this implies the boundedness of \( f)\) note that

\[
\Runderline{\int}_{A} f \text{ and } \Roverline{\int}_{A} f
\]
are always defined in \(E^{*}\).

Below, we always restrict \((f)\) to a fixed \((A \in \mathcal{C})\) (or \((A \in \mathcal{C}_{s})\)); \(\mathcal{P}, \mathcal{P}', \mathcal{P}''\) and \(\mathcal{P}^*\) denote \(\mathcal{C}\)-partitions of \((A,\cdot)\).

We now obtain the following result for any additive \((m : \mathcal{C} \rightarrow [0, \infty))\).

Corollary \(\PageIndex{1}\)

If \((\mathcal{P}')\) refines \((\mathcal{P}'')\) \((\S1)\), then

\[
\underline{S}(f, \mathcal{P}') \leq \underline{S}(f, \mathcal{P}) \leq \overline{S}(f, \mathcal{P}) \leq \overline{S}(f, \mathcal{P}').
\]

Proof

Let \((\mathcal{P}') = \{A_i\}, \mathcal{P} = \{B_{i,k}\}\) and

\[
(\forall i) \quad A_i = \bigcup_{k} B_{i,k}.
\]

By additivity,

\[
[m A_i = \sum_k m B_{i,k}]
\]

Also, \((B_{i,k} \subseteq A_i)\) implies

\[
[\sup f[B_{i,k}] \leq \sup f[A_i]; \ \inf f[B_{i,k}] \geq \inf f[A_i].
\]

So setting

\[
a_i = \inf f[A_i] \text{ and } b_{i,k} = \inf f[B_{i,k}],
\]

we get

\[
[m A_i = \sum_k m B_{i,k}]
\]

Similarly,

\[
[\overline{S}(f, \mathcal{P}') \leq \overline{S}(f, \mathcal{P})]
\]

and

\[
[\overline{S}(f, \mathcal{P}) \leq \overline{S}(f, \mathcal{P}')]
\]

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For any \( \mathcal{P}' \) and \( \mathcal{P}'' \),
\[
\underline{S}(f, \mathcal{P}') \leq \overline{S}(f, \mathcal{P}'').
\]
Hence
\[
R \underline{\int}_A f \leq R \overline{\int}_A f.
\]

Proof

Let \( \mathcal{P} = \mathcal{P}' \cap \mathcal{P}'' \) (see §1). As \( \mathcal{P} \) refines both \( \mathcal{P}' \) and \( \mathcal{P}'' \), Corollary 1 yields
\[
\underline{S}(f, \mathcal{P}') \leq \underline{S}(f, \mathcal{P}) \leq \overline{S}(f, \mathcal{P}) \leq \overline{S}(f, \mathcal{P}'').
\]
Thus, indeed, no lower sum \( \underline{S}(f, \mathcal{P}') \) exceeds any upper sum \( \overline{S}(f, \mathcal{P}'') \).

Hence also,
\[
\sup_{\mathcal{P}'} \underline{S}(f, \mathcal{P}') \leq \inf_{\mathcal{P}''} \overline{S}(f, \mathcal{P}''),
\]
i.e.,
\[
R \underline{\int}_A f \leq R \overline{\int}_A f,
\]
as claimed.

Lemma

A map \( f : A \rightarrow E^1 \) is \( R \)-integrable iff \( f \) is bounded and, moreover,
\[
(\forall \varepsilon > 0) \ (\exists \mathcal{P}) \ (\overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < \varepsilon).
\]

Proof

By formulas (1) and (2),
\[
\underline{S}(f, \mathcal{P}) \leq R \underline{\int}_A f \leq R \overline{\int}_A f \leq \overline{S}(f, \mathcal{P}).
\]
Hence (3) implies
\[
|\left| R \overline{\int}_A f - R \underline{\int}_A f \right| < \varepsilon.
\]
As \(\varepsilon\) is arbitrary, we get
\[
R \overline{\int}_A f = R \underline{\int}_A f;
\]
so \(f\) is R-integrable.

Conversely, if so, definitions (b) and (c) imply the existence of \(\mathcal{P}'\) and \(\mathcal{P}''\) such that
\[
\underline{S}(f, \mathcal{P}') > R \int_A f - \frac{1}{2} \varepsilon\]
and
\[
\overline{S}(f, \mathcal{P}'') < R \int_A f + \frac{1}{2} \varepsilon.
\]
Let \(\mathcal{P}\) refine both \(\mathcal{P}'\) and \(\mathcal{P}''\). Then by Corollary 1,
\[
\begin{aligned}
\overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) & \leq \overline{S}(f, \mathcal{P}'') - \underline{S}(f, \mathcal{P}') \\
& < \left( R \int_A f + \frac{1}{2} \varepsilon \right) - \left( R \int_A f - \frac{1}{2} \varepsilon \right) = \varepsilon,
\end{aligned}
\]
as required. \(\square\)

Lemma \(\PageIndex{2}\)

Let \(f\) be \(\mathcal{C}\)-simple; say, \(f = a_i \chi_{A_i}\) for some \(\mathcal{C}\)-partition \(\mathcal{P}^* = \{A_i\}\) of \(A\) (we then write
\[
f = \sum_i a_i \chi_{A_i}\]
on \(A\); see Note 4 of §4).

Then
\[
\begin{aligned}
R \underline{\int}_A f = R \overline{\int}_A f = \underline{S}(f, \mathcal{P}^*) = \overline{S}(f, \mathcal{P}^*) = \sum_i a_i m_{A_i}.
\end{aligned}
\]
Hence any finite \(\mathcal{C}\)-simple function is R-integrable, with \(R \int_A f\) as in (4).
Proof

Given any \(\mathcal{C}\)-partition \(\mathcal{P} = \{B_k\}\) of \(A\), consider
\[
\mathcal{P}^* \cdot \mathcal{P} = \{A_i \cap B_k\}.
\]
As \(f = \inf_{A_i} \cap B_k\) on \(A_i \cap B_k\) (even on all of \(A_i\)),
\[
\inf_{A_i \cap B_k} f = \sup_{A_i \cap B_k} f.
\]
Also,
\[
A = \bigcup_{i, k} \left( A_i \cap B_k \right) \quad \text{(disjoint)}
\]
and
\[
\forall i \quad A_i = \bigcup_{k} \left( A_i \cap B_k \right) \text{ (disjoint)}
\]
so
\[
\mu A_i = \sum_k m \left( A_i \cap B_k \right)
\]
and
\[
\underline{S}(f, \mathcal{P}) = \sum_i \sum_k a_i m \left( A_i \cap B_k \right) = \sum_i a_i m A_i = \underline{S}(f, \mathcal{P}^*)
\]
for any such \(\mathcal{P}\).

Hence also
\[
\sum_i a_i m A_i = \sup_{\mathcal{P}} \underline{S}(f, \mathcal{P}) = \int_A f.
\]
Similarly for \(\overline{\int}_A f\). This proves (4).

If, further, \((f)\) is finite, it is bounded (by \(\max |a_i|\)) since there are only finitely many \(a_i\); so \((f)\) is R-integrable on \(A\), and all is proved. \(\square\)

Note 1. Thus \(\underline{\int} S\) and \(\overline{\int} S\) are integrals of \(\mathcal{C}\)-simple maps, and definition (b) can be restated:
\[
\underline{\int}_A f = \sup \{ g \mid \text{ for all } \mathcal{P}, g \text{ is } \mathcal{C}\text{-simple with } g \leq f \}
\]
\[
\overline{\int}_A f = \inf \{ h \mid \text{ for all } \mathcal{P}, h \text{ is } \mathcal{C}\text{-simple with } h \geq f \}
\]
taking the sup and inf over all \(\mathcal{C}\)-simple maps \((g, h)\) with
\[
g \leq f \leq h \text{ on } A.
\]
(Verify by properties of glb and lub!)
Therefore, we can now develop R-integration as in §§4-5, replacing elementary maps by \(\text{C}\)-simple maps, with \(S=E^n\). In particular, Problem 5 in §5 works out as before.

Hence linearity (Theorem 1 of §6) follows, with the same proof. One also obtains additivity (limited to \(\text{C}\)-partitions). Moreover, the R-integrability of \(f\) and \(g\) implies that of \(fg\), \(\vee g\), \(\wedge g\), and \(|f|\) (See the Problems.)

Theorem \(\PageIndex{1}\)

If \(f_i \rightarrow f\) (uniformly) on \(A\) and if the \(f_i\) are R-integrable on \(A\), so also is \(f\). Moreover,

\[
\lim_{i \to \infty} R \int A |f-f_i| = 0 \quad \text{and} \quad \lim_{i \to \infty} R \int A f_i = R \int A f.
\]

Proof

As all \(f_i\) are bounded (definition (c)), so is \(f\) by Problem 10 of Chapter 4, §12.

Now, given \(\varepsilon>0\), fix \(k\) such that

\[
(\forall i \geq k) \quad |f-f_i| < \frac{\varepsilon}{mA} \quad \text{on} \quad A.
\]

Verify that

\[
(\forall i \geq k) \quad (\forall \mathcal{P}) \quad |\underline{S}(f-f_i, \mathcal{P})| < \varepsilon \quad \text{and} \quad |\overline{S}(f-f_i, \mathcal{P})| < \varepsilon;
\]

fix one such \(f_i\) and choose a \(\mathcal{P}\) such that

\[
\overline{S}(f_i, \mathcal{P}) - \underline{S}(f_i, \mathcal{P}) < \varepsilon,
\]

which one can do by Lemma 1. Then for this \(\mathcal{P}\),

\[
\overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < 3 \varepsilon.
\]

(Why?) By Lemma 1, then, \(f\) is R-integrable on \(A\).

Finally,

\[
|R \int A f - R \int A f_i| \leq R \int A |f-f_i| \leq R \int A \left(\frac{\varepsilon}{mA}\right) = mA \left(\frac{\varepsilon}{mA}\right) = \varepsilon
\]

for all \(i \geq k\). Hence the second clause of our theorem follows, too. \(\square\)

Corollary \(\PageIndex{3}\)
If \( f : E^1 \rightarrow E^1 \) is bounded and regulated (Chapter 5, §10) on \( A=[a, b] \), then \( f \) is R-integrable on \( A \).

In particular, this applies if \( f \) is monotone, or of bounded variation, or relatively continuous, or a step function, on \( A \).

**Proof**

By Lemma 2, this applies to \( \mathcal{C} \)-simple maps.

Now, let \( f \) be regulated (e.g., of the kind specified above).

Then by Lemma 2 of Chapter 5, §10,

\[
\lim_{i \to \infty} g_i \quad \text{(uniformly)}
\]

for finite \( \mathcal{C} \)-simple \( g_i \).

Thus \( f \) is R-integrable on \( A \) by Theorem 1.

II. Henceforth, we assume that \( m \) is a measure on a \( \sigma \)-ring \( \mathcal{M} \supseteq \mathcal{C} \) in \( E^n \), with \( m<\infty \) on \( \mathcal{C} \). (For a reader who took the "limited approach," it is now time to consider §§4-6 in full.) The measure \( m \) may, but need not, be Lebesgue measure in \( E^n \).

Theorem \( \PageIndex{2} \)

If \( f : E^n \rightarrow E^1 \) is R-integrable on \( A \in \mathcal{C} \), it is also Lebesgue integrable (with respect to \( m \) as above) on \( A \) and

\[
L \int_A f = R \int_A f.
\]

**Proof**

Given a \( \mathcal{C} \)-partition \( \mathcal{P} = \{ A_i \} \) of \( A \), define the \( \mathcal{C} \)-simple maps

\[
g = \sum_i a_i C_{A_i} \quad \text{and} \quad h = \sum_i b_i C_{A_i},
\]

with

\[
a_i = \inf f|_{A_i} \quad \text{and} \quad b_i = \sup f|_{A_i}.
\]

Then \( g \leq f \leq h \) on \( A \) with

\[
\underline{S}(f, \mathcal{P}) = \sum_i a_i m A_i = L \int_A g
\]

and

\[
\overline{S}(f, \mathcal{P}) = \sum_i b_i m A_i = L \int_A h.
\]
By Theorem 1(c) in §5,
\[
\underline{S}(f, \mathcal{P}) = L \int_{A} g \leq L \underline{\int}_{A} f \leq L \overline{\int}_{A} f \leq L \int_{A} h = \overline{S}(f, \mathcal{P}).
\]
As this holds for any \(\mathcal{P}\), we get
\[
R \underline{\int}_{A} f = \sup_{\mathcal{P}} \underline{S}(f, \mathcal{P}) \leq L \underline{\int}_{A} f \leq L \overline{\int}_{A} f = \inf_{\mathcal{P}} \overline{S}(f, \mathcal{P}) = R \overline{\int}_{A} f.
\]
But by assumption,
\[
(R \underline{\int}_{A} f) = R \overline{\int}_{A} f.
\]
Thus these inequalities become equations:
\[
(R \int_{A} f) = \underline{\int}_{A} f = \overline{\int}_{A} f = R \int_{A} f.
\]
Also, by definition (c), \(\mathcal{A}\) is bounded on \(\mathcal{A}\); so \(|f| < K < \infty\) on \(\mathcal{A}\). Hence
\[
|\int_{A} f| \leq \int_{A}|f| \leq K \cdot m A < \infty.
\]
Thus
\[
\underline{\int}_{A} f = \overline{\int}_{A} f \neq \pm \infty,
\]
i.e., \(f\) is Lebesgue integrable, and
\[
L \int_{A} f = R \int_{A} f,
\]
as claimed. \(\Box\)

**Note 2.** The converse fails. For example, as shown in the example in §4, \(\mathcal{C}\) (\(\mathcal{C}\)) is L-integrable on \((A = [0,1])\).

Yet \(f\) is not \((R)\)-integrable.

For \(\mathcal{C}\)-partitions involve intervals containing both rationals (on which \(f=1\)) and irrationals (on which \(f=0\)). Thus for any \(\mathcal{P}\),
\[
\underline{S}(f, \mathcal{P}) = 0 \text{ and } \overline{S}(f, \mathcal{P}) = 1 \cdot m A = 1.
\]
(Why?) So
\[
(R \overline{\int}_{A} f) = \inf \overline{S}(f, \mathcal{P}) = 1,\]

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while

\[ R \underline{\int}_{A} f = 0 \neq R \underline{\int}_{A} f. \]

**Note 3.** By Theorem 1, any \( R \int_{A} f \) is also a Lebesgue integral. Thus the rules of §§5-6 apply to R-integrals, provided that the functions involved are R-integrable. For a deeper study, we need a few more ideas.

Definitions (continued)

(d) The mesh \( |\mathcal{P}| \) of a \( \mathcal{C} \)-partition \( \mathcal{P} = \{A_1, \ldots, A_q\} \) is the largest of the diagonals \( d A_i \):

\[ |\mathcal{P}| = \max \{d A_1, d A_2, \ldots, d A_q\}. \]

**Note 4.** For any \( A \in \mathcal{C} \), there is a sequence of \( \mathcal{C} \)-partitions \( \mathcal{P}_k \) such that

(i) each \( \mathcal{P}_{k+1} \) refines \( \mathcal{P}_k \), and

(ii) \( \lim_{k \to \infty} |\mathcal{P}_k| = 0 \).

To construct such a sequence, bisect the edges of \( A \) so as to obtain \( 2^n \) subintervals of diagonal \( dA \) (Chapter 3, §7). Repeat this with each of the subintervals, and so on. Then

\[ |\mathcal{P}_k| = \frac{dA}{2^k} \to 0. \]

**Lemma \( \PageIndex{3} \)**

Let \( f : A \to E^1 \) be bounded. Let \( \mathcal{P}_k \) satisfy (i) of Note 4. If \( \mathcal{P}_k = \{A_1, \ldots, A_q\} \) put

\[ g_k = \sum_{i=1}^{q} C_{A_i} \inf f[A_i] \]

and

\[ h_k = \sum_{i=1}^{q} C_{A_i} \sup f[A_i]. \]

Then the functions

\[ g = \sup_k g_k \quad \text{and} \quad h = \inf_k h_k \]

are Lebesgue integrable on \( A \) and

\[ R \underline{\int}_{A} g = \lim_{k \to \infty} |\mathcal{P}_k| \sup \left\{ f \left( \mathcal{P}_k \right) \right\} \leq R \underline{\int}_{A} f \leq R \underline{\int}_{A} h. \]
Proof

As in Theorem 2, we obtain \( g_k \leq f \leq h_k \) on \( A \) with
\[
\int_A g_k = \underline{S}(f, \mathcal{P}_k)
\]
and
\[
\int_A h_k = \overline{S}(f, \mathcal{P}_k).
\]
Since \( \mathcal{P}_{k+1} \) refines \( \mathcal{P}_k \), it also easily follows that
\[
\begin{align*}
g_k & \leq g_{k+1} \leq \sup_k g_k = g \leq f \leq h = \inf_k h_k \leq h_{k+1} \leq h_k.
\end{align*}
\]
(Verify!)

Thus \( \int_A g_k \rightleftharpoons \frac{\text{uparrow}}{\text{downarrow}} \) and \( \int_A h_k \rightleftharpoons \frac{\text{uparrow}}{\text{downarrow}} \) and so
\[
\begin{align*}
g & = \sup_k g_k = \lim_k g_k \rightarrow \infty, \\
h & = \inf_k h_k = \lim_k h_k \rightarrow \infty.
\end{align*}
\]

Also, as \( f \) is bounded
\[
\int_A (g_k \rightleftharpoons \frac{\text{uparrow}}{\text{downarrow}} K \text{ on } A.)
\]
The definition of \( g_k \) and \( h_k \) then implies
\[
\begin{align*}
(\forall k) \quad |g_k| & \leq K \text{ and } |h_k| \leq K \text{ (why?),}
\end{align*}
\]
with
\[
\int_A (K) = K \cdot m_A < \infty.
\]
The \( g_k \) and \( h_k \) are measurable (even simple) on \( A \), with \( g_k \rightarrow g \) and \( h_k \rightarrow h \).

Thus by Theorem 5 and Note 1, both from §6, \( g \) and \( h \) are Lebesgue integrable, with
\[
\begin{align*}
\int_A g & = \lim_k g_k \rightarrow \infty, \\
\int_A h & = \lim_k h_k \rightarrow \infty.
\end{align*}
\]

As
\[
\int_A g_k \rightleftharpoons \underline{S}(f, \mathcal{P}_k) \rightleftharpoons \underline{\int}_A f
\]
and

\[
\int_A h_k \rightleftharpoons \overline{S}(f, \mathcal{P}_k) \rightleftharpoons \overline{\int}_A f
\]
\[
\int_{A} h_{k} = \overline{S}(f, \mathcal{P}_{k}) \geq R \int_{A} f \]

Passage to the limit in equalities yields (6). Thus the lemma is proved. \(\square\)

**Lemma \(\PageIndex{4}\)**

With all as in Lemma 3, let \(A\) be the union of the boundaries of all intervals from all \(\mathcal{P}_{k}\). Let \(f\) be continuous on \(A\). Then the following.

(i) If \(f\) is continuous at \(p \in A\), then \(h(p) = g(p)\).

(ii) The converse holds if \(p \not\in A\).

**Proof**

For each \((k, p)\) is in one of the intervals in \(\mathcal{P}_{k}\); call it \((A_{kp})\).

If \(p \in A-B\), \(p\) is an interior point of \((A_{kp})\); so there is a globe \(G_{p}(\delta_{k}) \subseteq A_{kp}\).

Also, by the definition of \(g_{k}\) and \(h_{k}\),

\[
g_{k}(p) = \inf f[A_{kp}] \text{ and } h_{k} = \sup f[A_{kp}].
\]

(Why?)

Now fix \(\epsilon > 0\). If \(g(p) = h(p)\), then

\[
0 = h(p) - g(p) = \lim_{k \to \infty} (h_{k}(p) - g_{k}(p))
\]

so

\[
(\exists k) \quad |h_{k}(p) - g_{k}(p)| = \sup f[A_{kp}] - \inf f[A_{kp}] < \epsilon.
\]

As \(G_{p}(\delta_{k}) \subseteq A_{kp}\), we get

\[
\forall x \in G_{p}(\delta_{k}) \quad |f(x) - f(p)| < \epsilon.
\]

proving continuity (clause (ii)).

For (i), given \(\epsilon > 0\), choose \(\delta > 0\) so that

\[
\forall x, y \in A \cap G_{p}(\delta) \quad |f(x) - f(y)| < \epsilon.
\]
Because

$$\left(\forall \delta>0\right) \exists k_{0} \left(\forall k>k_{0}\right) \left|\mathcal{P}_{k}\right|<\delta$$

for \(k>k_{0}\), \(A_{k p} \subseteq G_{p}(\delta)\). Deduce that

$$\left(\forall k>k_{0}\right) \left|h_{k}(p)-g_{k}(p)\right| \leq \varepsilon. \quad \square$$

Note 5. The Lebesgue measure of \(\langle B \rangle\) in Lemma 4 is zero; for \(\langle B \rangle\) consists of countably many "faces" (degenerate intervals), each of measure zero.

Theorem \(\PageIndex{3}\)

A map \(f: A \rightarrow E^{1}\) is R-integrable on \(\langle A \rangle\) (with \(\langle m=\rangle\) Lebesgue measure) iff \(\langle f \rangle\) is bounded on \(\langle A \rangle\) and continuous on \(\langle A-Q \rangle\) for some \(\langle Q \rangle\) with \(\langle m Q=0 \rangle\).

Note that relative continuity on \(\langle A-Q \rangle\) is not enough—take \(\langle f=C_{\langle R \rangle} \rangle\) of Note 2.

Proof

If these conditions hold, choose \(\langle \left(\left(\text{int}\left\{ A \right\} =\text{int}\left\{ A \right\}\right) \rangle\) as in Lemma 4.

Then by the assumed continuity, \(\langle g=h \rangle\) on \(\langle A-Q, m Q=0 \rangle\).

Thus

$$\langle \text{int}\left\{ A \right\} g=\text{int}\left\{ A \right\} h \rangle$$

(Corollary 2 in §5).

Hence by formula (6), \(\langle f \rangle\) is R-integrable on \(\langle A \rangle\).

Conversely, if so, use Lemma 1 with

$$\langle \varepsilon=1, \frac{1}{2}, \ldots, \frac{1}{k}, \ldots \rangle$$

to get for each \(\langle k \rangle\) some \(\langle \text{mathcal}\{P\}_{k} \rangle\) such that

$$\langle \overline{S}(f, \text{mathcal}\{P\}_{k})-\underline{S}(f, \text{mathcal}\{P\}_{k})<\frac{1}{k} \rightarrow 0 \rangle$$

By Corollary 1, this will hold if we refine each \(\langle \text{mathcal}\{P\}_{k} \rangle\) step by step, so as to achieve properties (i) and (ii) of Note 4 as well. Then Lemmas 3 and 4 apply.

As

$$\langle \overline{S}(f, \text{mathcal}\{P\}_{k})-\underline{S}(f, \text{mathcal}\{P\}_{k}) \rangle$$

---------------------------------
formula (6) show that
\[
\int_{A} g = \lim_{k \to \infty} \underline{S}(f, \mathcal{P}_{k}) = \lim_{k \to \infty} \overline{S}(f, \mathcal{P}_{k}) = \int_{A} h.
\]
As \((h')\) and \((g')\) are integrable on \((A)\),
\[
\int_{A} (h-g) = \int_{A} h - \int_{A} g = 0.
\]
Also \((h-g \geq 0;\) so by Theorem 1(h) in §5, \((h=g)\) on \((A-Q^{'prime})\), \(m Q^{'prime}=0\) (under Lebesgue measure). Hence by Lemma 4, \((f)\) is continuous on
\[
[A-Q^{'prime}-B,]
\]
with \((mB=0;\) (Note 5).
Let \((Q=Q^{'prime} \cup B;\) Then \((m Q=0;)\) and
\[
[A-Q=A-Q^{'prime}-B;]
\]
so \((f)\) is continuous on \((A-Q;)\) This completes the proof.\(\square\)

**Note 6.** The first part of the proof does not involve \((B;)\) and thus works even if \((m;)\) is not the Lebesgue measure. The second part requires that \((mB=0;\).

Theorem 3 shows that R-integrals are limited to a.e. continuous functions and hence are less flexible than L-integrals: Fewer functions are R-integrable, and convergence theorems (§6, Theorems 4 and 5) fail unless \((R \int_{A} f)\) exists.

**III. Functions** \(f : E^{n} \to E^{s}\) For such functions, R-integrals are defined componentwise (see §7). Thus \((f=\left\langle f_{1}, \ldots, f_{s}\right\rangle;\) \(\Rightarrow d_{h},\) \(\left\langle f_{i}\right\rangle;\) \(\Rightarrow d_{h}\)) is R-integrable on \((A;\) iff all \((f_{k};\) \(\Rightarrow d_{k}\)) are, and then
\[
[R \int_{A} f = \sum_{k=1}^{s} \\overline{e}_{k} R \int_{A} f_{k}.\]
\]
A complex function \((f)\) is R-integrable iff \((f_{1}(re));\) and \((f_{1}(im));\) are, and then
\[
[R \int_{A} f = R \int_{A} f_{re} + i R \int_{A} f_{im}.\]
\]
Via components, Theorems 1 to 3, Corollaries 3 and 4, additivity, linearity, etc., apply.

**IV. Stieltjes Integrals.** Riemann used Lebesgue premeasure \((\nu;)\) only. But as we saw, his method admits other premeasures, too.

Thus in \((E^{1};\) we may let \((\nu;\) be the \((LS;)\) premeasure \(\left\langle s_{(\alpha)}\right\rangle\) or the \((LS;)\) measure \(m_{\left\langle \alpha_{\downarrow} \right\rangle}\) where \((\alpha;\) \(\Rightarrow d_{1}\)) (Chapter 7, §5, Example (b), and Chapter 7, §9).
Then
\[
\int_{A}^{B} f \, dm
\]
is called the Riemann-Stieltjes (RS) integral of \(f\) with respect to \(\alpha\), also written
\[
\int_{A}^{B} f \, d\alpha \quad \text{or} \quad \int_{a}^{b} f(x) \, d\alpha(x)
\]
(the latter if \(A=[a, b]\)); \(f\) and \(\alpha\) are called the integrand and integrator, respectively.

If \(\alpha(x)=x\), \(m_{\alpha}\) becomes the Lebesgue measure, and
\[
\int f(x) \, d\alpha(x)
\]
turns into
\[
\int f(x) \, dx.
\]
Our theory still remains valid; only Theorem 3 now reads as follows.

Corollary \(\PageIndex{4}\)

If \(f\) is bounded and a.e. continuous on \(A=[a, b]\) (under an LS measure \(m_{\alpha}\)) then
\[
\int_{a}^{b} f \, d\alpha
\]
exists. The converse holds if \(\alpha\) is continuous on \(A\).

For by Notes 5 and 6, the "only if" in Theorem 3 holds if \((m_{\alpha} B=0)\). Here consists of countably many endpoints of partition subintervals. But (see Chapter §9) \((m_{\alpha}\{p\}=0)\) if \(\alpha\) is continuous at \(p\). Thus the later implies \((m_{\alpha} B=0)\).

RS-integration has been used in many fields (e.g., probability theory, physics, etc.), but it is superseded by LS-integration, i.e., Lebesgue integration with respect to \(m_{\alpha}\) which is fully covered by the general theory of §§1-8.

Actually, Stieltjes himself used somewhat different definitions (see Problems 10-13), which amount to applying the set function \(\sigma_{\alpha}\) of Problem 9 in Chapter 7, §4, instead of \(s_{\alpha}\) or \(m_{\alpha}\). We reserve the name "Stieltjes integrals," denoted
\[
\int_{a}^{b} f \, d\alpha
\]
for such integrals, and "RS-integrals" for those based on \(m_{\alpha}\) or \(s_{\alpha}\) (this terminology is not standard).

Observe that \(\sigma_{\alpha}\) need not be \(\geq 0\). Thus for the first time, we encounter integration with respect to sign-changing set functions. A much more general theory is presented in §10 (see Problem 10 there).