8.11: The Radon–Nikodym Theorem. Lebesgue Decomposition

I. As you know, the indefinite integral

\[ \int f \, dm \]

is a generalized measure. We now seek conditions under which a given generalized measure \( \mu \) can be represented as

\[ \mu = \int f \, dm \]

for some \( f \) (to be found). We start with two lemmas.

Lemma \( \PageIndex{1} \)

Let \( m, \mu : \mathcal{M} \rightarrow [0, \infty) \) be finite measures in \( S \). Suppose \( S \in \mathcal{M}, \mu S > 0 \) (i.e., \( \mu \not \equiv 0 \)) and \( \mu \) is \( m \)-continuous (Chapter 7, §11).

Then there is \( \delta > 0 \) and a set \( P \in \mathcal{M} \) such that \( m P > 0 \) and

\[ \mu X \geq \delta \cdot m(X \cap P). \]

Proof

As \( m \in \mathcal{M} \) and \( \mu S > 0 \), there is \( \delta > 0 \) such that

\[ \mu S \geq \delta \cdot m S > 0. \]

Fix such a \( \delta \) and define a signed measure (Lemma 2 of Chapter 7, §11)
\[\Phi = \mu - \delta m,\]

so that

\[\forall Y \in \mathcal{M} \quad \Phi Y = \mu Y - \delta \cdot m Y;\]

hence

\[\Phi S = \mu S - \delta \cdot m S > 0.\]

By Theorem 3 in Chapter 7, §11 (Hahn decomposition), there is a \(\Phi\)-positive set \(P \in \mathcal{M}\) with a \(\Phi\)-negative complement \((-P=S-P) \in \mathcal{M}\).

Clearly, \((m P>0;)\) for if \((m P=0;\) the \((m)-\)continuity of \((\mu)\) would imply \((\mu P=0;)\), hence

\[\Phi P = \mu P - \delta \cdot m P = 0,\]

contrary to \(\Phi P \geq \Phi S > 0;\).

Also, \((P \supseteq Y)\) and \((Y \in \mathcal{M})\) implies \((\Phi Y \geq 0;\) so by (1),

\[0 \leq \mu Y - \delta \cdot m Y.\]

Taking \((Y=X \cap P;)\) we get

\[\delta \cdot m(X \cap P) \leq \mu(X \cap P) \leq \mu X,\]

as required. \(\square\)

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**Lemma \(\PageIndex{2}\)**

With \((m, \mu, S)\) as in Lemma 1, let \((\mathcal{H})\) be the set of all maps \((g : S \to E^*; \mathcal{M})\)-measurable and nonnegative on \((S;)\) such that

\[\int_X g dm \leq \mu X\]

for every set \((X;)\) from \((\mathcal{M})\).

Then there is \((f)\) in \((\mathcal{H})\) with

\[\int_S f dm = \max \{g \in \mathcal{M} \mid f \leq g \leq \mu X\}\]

**Proof**

\((\mathcal{H})\) is not empty; e.g., \((g=0;)\) is in \((\mathcal{H})\). We now show that...
Indeed, $(\forall g, h \in \mathcal{H}) \quad g \vee h = \max (g, h) \in \mathcal{H}$.\]

Now, given $(X \in \mathcal{M},)$ let $(Y=X(g>h))$ and $(Z=X(g \leq h).)$ Dropping "$\mu dm"$ for brevity, we have
\[
\int_{X}(g \vee h) = \int_{Y}(g \vee h) + \int_{Z}(g \vee h) = \int_{Y} g + \int_{Z} h \leq \mu Y + \mu Z = \mu X,\]
proving (2).

Let $k = \sup \{ g \in \mathcal{H} \} \int_{S} g \ dm \in E^*$.\]

Proceeding as in Problem 13 of Chapter 7, §6, and using (2), one easily finds a sequence $(g_n \uparrow, g_n \in \mathcal{H},)$ such that
\[
\lim_{n \rightarrow \infty} \int_{S} g_n dm = k.\]
(Verify!) Set
\[
f = \lim_{n \rightarrow \infty} g_n.\]
(It exists since $(\forall g_n \uparrow) \ By \ Theorem \ 4 \ in \ §6,$
\[
k = \lim_{n \rightarrow \infty} \int_{S} g_n = \int_{S} f.\]
Also, $(f)$ is $(\forall g \in \mathcal{H})$-measurable and $(\forall g \geq 0)$ on $(S,)$ as all $(g_n)$ are; and if $(X \in \mathcal{M},)$ then
\[
(f \forall n) \quad \int_{X} g_n \leq \mu X;\]
hence
\[
\int_{X} f = \lim_{n \rightarrow \infty} \int_{X} g_n \leq \mu X.\]
Thus $(f \in \mathcal{H})$ and
\[
\int_{S} f = k = \sup \{ g \in \mathcal{H} \} \int_{S} g,\]
i.e.,
\[
\int_{S} f = \max \{ g \in \mathcal{H} \} \int_{S} g \ \mu < \text{infty}.\]
This completes the proof.\]

**Note 1.** As $(\mu < \text{infty})$ and $(f \geq 0,)$ Corollary 1 in §5 shows that $(f)$ can be made finite on all of $(S,)$ Also, $(f)$ is
\(m\)-integrable on \(S\).

Theorem \(\PageIndex{1}\) (Radon-Nikodym)

If \((S, \mathcal{M}, m)\) is a \(\sigma\)-finite measure space, if \(S \in \mathcal{M}\), and if
\[
\mu : \mathcal{M} \rightarrow E^n(C^n)
\]
is a generalized \(m\)-continuous measure, then
\[
\mu = \int f \, dm \text{ on } \mathcal{M}
\]
for at least one map
\[
f : S \rightarrow E^n(C^n),
\]
\(\mathcal{M}\)-measurable on \(S\).

Moreover, if \(h\) is another such map, then \(m S (f \neq h) = 0\).

The last part of Theorem 1 means that \(f\) is "essentially unique." We call \(f\) the Radon-Nikodym \((RN)\) derivative of \(\mu\) with respect to \(m\).

**Proof**

Via components (Theorem 5 in Chapter 7, §11), all reduces to the case
\[
\mu : \mathcal{M} \rightarrow E\{n\} \leftarrow (C^n)\]

Then Theorem 4 (Jordan decomposition) in Chapter 7, §11, yields
\[
\mu = \mu^+ - \mu^-
\]
where \(\mu^+\) and \(\mu^-\) are finite measures \(\mu(0, \leq \infty)\) both \(m\)-continuous (Corollary 3 from Chapter 7, §11). Therefore, all reduces to the case \(0 \leq \mu < \infty\).

Suppose first that \(m\) too, is finite. Then if \(\mu = 0\), just take \(f = 0\).

If, however, \(\mu S > 0\), take \(f \in \mathcal{H}\) as in Lemma 2 and Note 1; \(f\) is nonnegative, bounded, and \(\mathcal{M}\)-measurable on \(S\),
\[
\int f \, dm \leq k = \sup \{ g \in \mathcal{M}\} \int g \, dm.
\]

and
\[
\int \{S \} f \, dm = k = \sup \{ g \in \mathcal{M}\} \int \{S \} g \, dm.
\]
We claim that \( f \) is the required map.

Indeed, let

\[ \nu = \mu - \int f \, dm; \]

so \( \nu \) is a finite \((m)\)-continuous measure \((\geq 0)\) on \((\mathcal{M})\) (Why?) We must show that \( \nu = 0 \).

Seeking a contradiction, suppose \( \nu > 0 \) Then by Lemma 1, there are \( P \in \mathcal{M} \) and \( \delta > 0 \) such that

\[ \nu \cap P > 0 \]

and

\[ \forall X \in \mathcal{M}, \quad \nu X \geq \delta \cdot m(X \cap P). \]

Now let

\[ g = f + \delta \cdot C_P; \]

so \( g \) is \((\mathcal{M})\)-measurable and \((\geq 0)\) Also,

\[ \int_X g = \int_X f + \delta \cdot m(X \cap P) \leq \int_X f + \nu(X \cap P) \leq \int_X f + \mu X \]

by our choice of \( \delta \) and \( \nu \). Thus \( g \in \mathcal{H} \). On the other hand,

\[ \int_S g = \int_S f + \delta \cdot m(P) > \int_S f + \mu P \]

contrary to

\[ k = \sup \{ g \in \mathcal{H} \} \]

This proves that \( \int f = \mu \) indeed.

Now suppose there is another map \( h \in \mathcal{M} \) with

\[ \mu = \int h \, dm \neq \infty \]

so

\[ \int(f-h) \, dm = 0. \]

(Why?) Let

\[ Y = S(f \geq h) \text{ and } Z = S(f < h); \]

so \( Y, Z \in \mathcal{M} \) (Theorem 3 of §2) and \( f-h \) is sign-constant on \( Y \) and \( Z \). Also, by construction,
\[ \int_{Y} (f-h) \, dm = 0 = \int_{Z} (f-h) \, dm. \]

Thus by Theorem 1(h) in §5, \( (f-h=0) \) a.e. on \( (Y,\nu) \) on \( (Z,\nu) \) and hence on \( (S=Y\cup Z) \) that is,

\[ [mS(f\neq h)=0]. \]

Thus all is proved for the case \( mS<\infty \).

Next, let \( (m) \) be \( (\sigma) \)-finite:

\[ S=\bigcup_{k=1}^{\infty} S_{k} \text{ (disjoint)} \]

for some sets \( (S_{k}) \in \mathcal{M} \) with \( mS_{k}<\infty \).

By what was shown above, on each \( (S_{k}) \) there is an \( \mathcal{M} \)-measurable map \( (f_{k} \geq 0) \) such that

\[ \int_{X} f_{k} \, dm = \mu X \]

for all \( (\mathcal{M}) \)-sets \( (X \subseteq S_{k}) \). Fixing such an \( (f_{k}) \) for each \( (k) \) define \( (f: S \rightarrow E^{1}) \) by

\[ f=f_{k} \text{ on } S_{k}, \quad k=1,2, \ldots. \]

Then (Corollary 3 in §1) \( (f) \) is \( (\mathcal{M}) \)-measurable and \( (f \geq 0) \) on \( (S) \).

Taking any \( (X \subseteq \mathcal{M}) \) set \( (X_{k}=X \cap S_{k}) \) Then

\[ X=\bigcup_{k=1}^{\infty} X_{k} \text{ (disjoint)} \]

and \( (X_{k}) \in \mathcal{M} \). Also,

\[ \int (\forall k) \int_{X_{k}} f d m = \int_{X} f d m = \mu X \]

Thus by \( (\sigma) \)-additivity (Theorem 2 in §5),

\[ \int \int_{X_{k}} f d m = \sum_{k=1}^{\infty} \int_{X_{k}} f d m = \mu X < \infty \text{ quad (} \mu \text{ is finite!)}. \]

Thus \( (f) \) is as required, and its "uniqueness" follows as before. \( \Box \)

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**Note 2.** By Definition 3 in §10, we may write

\[ ["d \mu=f \, dm"] \]

for
\[\int f \, dm = \mu.\]

**Note 3.** Using Definition 2 in §10 and an easy "componentwise" proof, one shows that Theorem 1 holds also with \((m)\) replaced by a generalized measure \((s)\). The formulas

\[\mu = \int f \, dm \text{ and } mS(f \neq h) = 0\]

then are replaced by

\[\mu = \int f \, ds \text{ and } v_{s}S(f \neq h) = 0.\]

**II.** Theorem 1 requires \((\mu)\) to be \((m)\)-continuous \((\mu \ll m)\). We want to generalize Theorem 1 so as to lift this restriction. First, we introduce a new concept.

**Definition**

Given two set functions \(s, t : \mathcal{M} \rightarrow E (\mathcal{M} \subseteq 2^{S})\) we say that \(s\) is \(t\)-singular \((s \perp t)\) iff there is a set \(P \in \mathcal{M}\) such that \(v_{t} P = 0\) and

\[(\forall X \in \mathcal{M}) \quad s X = s(X \cap P).\]

(We then briefly say "s resides in \((P)\")

For generalized measures, this means that

\[(\forall X \in \mathcal{M}) \quad s X = s(X \cap P).\]

**Why?**

**Corollary \(^{1}\)**

If the generalized measures \((s, u : \mathcal{M} \rightarrow E)\) are \((t)\)-singular, so is \((k s)\) for any scalar \((k)\) (if \((s)\) is scalar valued, \((k)\) may be a vector).

So also are \((s \pm u)\) provided \((t)\) is additive.

**Proof**

(Exercise! See Problem 3 below.)

**Corollary \(^{2}\)**

If a generalized measure \((s : \mathcal{M} \rightarrow E)\) is \((t)\)-continuous \((s \ll t)\) and also \((t)\)-singular \((s \perp t)\) then \((s = 0)\) on \((\mathcal{M})\).
Proof

As \(s \perp t\), formula (3) holds for some \((P \in \mathcal{M}, v_{t} \{t\} P=0)\). Hence for all \((X \in \mathcal{M}, X \subseteq P)\)

\[[s(X-P)=0 \text { (for } X-P \subseteq P \text{)}]\]

and

\[[v_{t}(X \cap P)=0 \text { (for } X \cap P \subseteq P \text{).}\]

As \(s \ll t\) we also have \((s(X \cap P)=0)\) by Definition 3(i) in Chapter 7, §11. Thus by additivity,

\[[sX=s(X \cap P)+s(X-P)=0,]\]

as claimed. \(\square\)

Theorem \(\PageIndex{2}\) (Lebesgue decomposition)

Let \(s, t : \mathcal{M} \to E\) be generalized measures.

If \(v_{s}\) is \(t\)-finite (Definition 3(iii) in Chapter 7, §11), there are generalized measures \(s', s'' : \mathcal{M} \to E\) such that

\[[s' \ll t \text { and } s'' \perp t\]

and

\[[s=s'+s''].\]

Proof

Let \((v_{\{0\}})\) be the restriction of \((v_{\{s\}})\) to

\[[\mathcal{M}_{\{0\}}=\{X \in \mathcal{M} | v_{t} X=0\}\].\]

As \((v_{\{s\}})\) is a measure (Theorem 1 of Chapter 7, §11), so is \((v_{\{0\}})\) (for \((\mathcal{M}_{\{0\}})\) is a \((\sigma)\)-ring; verify!).

Thus by Problem 13 in Chapter 7, §6, we fix \((P \in \mathcal{M}_{\{0\}},)\) with

\[[v_{\{s\}} P=v_{\{0\}} P=\max \{v_{\{s\}} X | X \in \mathcal{M}_{\{0\}}'\}].\]

As \((P \in \mathcal{M}_{\{0\}})\) we have \((v_{\{t\}} P=0)\) hence

\[[|sP| \leq v_{\{s\}} P<\infty].\]
Now define \(s', s'', v', v''\) by setting, for each \(X \in \mathcal{M}\),

\[
\begin{aligned}
    s' X &= s(X - P); \\
    s'' X &= s(X \cap P); \\
    v' X &= v_s(X - P); \\
    v'' X &= v_s(X \cap P).
\end{aligned}
\]

As \(s\) and \(v_s\) are \(\sigma\)-additive, so are \(s', s'', v', v''\). (Verify!) Thus \(s', s'' : \mathcal{M} \rightarrow E\) are generalized measures, while \(v'\) and \(v''\) are measures \((\geq 0)\).

We have

\[
\forall X \in \mathcal{M} \quad s X = s(X - P) + s(X \cap P) = s' X + s'' X;
\]

i.e.,

\[
s = s' + s''.
\]

Similarly one obtains \(v = v' + v''\).

Also, by (5), since \(X \cap P = \emptyset\),

\[-P \supseteq X \text{ and } X \in \mathcal{M} \Rightarrow s'' X = 0,
\]

while \((v_{t} P = 0)\) (see above). Thus \(s''\) is \((t)\)-singular, residing in \(\mathcal{P}\).

To prove \(s' \ll t\), it suffices to show that \(v' \ll t\) (by (4) and (6), \(v' X = 0\)) implies \(s' X = 0\).

Assume the opposite. Then

\[
\exists Y \in \mathcal{M} \quad v_{t} Y = 0,
\]

(i.e., \(Y \in \mathcal{M}_{0}\)), but

\[
0 < v' Y = v_{t} Y = 0.
\]

So by additivity,

\[
v' Y = 0 < v' Y = v_{t} Y = 0,
\]

with \((\exists Y \cup P \in \mathcal{M} \setminus \{0\})\) contrary to
This contradiction completes the proof. \(\square\)

**Note 4.** The set function \(s^\prime \perp t\) in Theorem 2 is bounded on \(\mathcal{M}\). Indeed, \(s^\prime \perp t\) yields a set \(P \in \mathcal{M}\) such that

\[(\forall X \in \mathcal{M}) \quad s^\prime (X-P)=0;\]

and \(\forall \{t \mid P=0\}\) implies \(\forall \{s \mid P<\infty\}\) (Why?) Hence

\[s^\prime \perp t\quad X=s^\prime \perp t\quad(X \cap P)+s^\prime \perp t\quad(X-P)=s^\prime \perp t\quad(X \cap P);\]

As \(s=s^\prime +s^\prime \perp t\) we have

\[\forall \{\text{left} \mid s^\prime \perp t\} \quad \text{right} \quad \leq \{s \mid \text{left} \mid s^\prime \perp t\} \quad \text{right} \quad \leq v \{s \} + v \{s^\prime \};\]

so

\[\forall \{\text{left} \mid s^\prime \perp t\} \quad X \cap P=\{\text{left} \mid s^\prime \perp t\} \quad(X \cap P) \quad \text{right} \quad \leq v \{s \} + v \{s^\prime \} \quad P;\]

But \(\forall \{s^\prime \}\) by \((t)\)-continuity (Theorem 2 of Chapter 7, §11). Thus \(\forall \{\text{left} \mid s^\prime \perp t\} \quad \text{right} \quad \leq v \{s \} \quad P<\infty\) on \(\mathcal{M}\).

**Note 5.** The Lebesgue decomposition \(s=s^\prime +s^\prime \perp t\) in Theorem 2 is unique. For if also

\[u^\prime \perp t\quad \text{and} \quad u^\prime \perp t\]

then with \(\mathcal{P}\) as in Problem 3, \((\forall X \in \mathcal{M})\)

\[s^\prime \perp t\quad(X \cap P)+s^\prime \perp t\quad(X \cap P)=u^\prime \perp t\quad(X \cap P)+u^\prime \perp t\quad(X \cap P);\]

and \(\forall \{t \mid (X \cap P)=0\}\) But

\[s^\prime \perp t\quad(X \cap P)=0=u^\prime \perp t\quad(X \cap P);\]

by \((t)\)-continuity; so (8) reduces to

\[s^\prime \perp t\quad(X \cap P)=u^\prime \perp t\quad(X \cap P);\]

or \(s^\prime \perp t\quad X=u^\prime \perp t\quad(X),\) (for \(s^\prime \perp t\) and \(u^\prime \perp t\) reside in \(\mathcal{P}\)). Thus

\(s^\prime \perp t\quad(X \cap P)=u^\prime \perp t\quad(X \cap P)\) on \(\mathcal{M}\).
By Note 4, we may cancel \(s' + s''\) and \(u' + u''\) in
\[
s' + s'' = u' + u''
\]
to obtain \(s' = u'\) also.

**Note 6.** If \((E = E^n, \{C^n\}, \{\cdot\})\) the \((t')\)-finiteness of \(v_s\) in Theorem 2 is redundant, for \(v_s\) is even bounded (Theorem 6 in Chapter 7, §11).

We now obtain the desired generalization of Theorem 1.

**Corollary 3**

If \((S, \mathcal{M}, m)\) is a \((\sigma)\)-finite measure space \((S \in \mathcal{M})\), then for any generalized measure
\[
\mu : \mathcal{M} \rightarrow E^n, \mathcal{N} \rightarrow C^n
\]
there is a unique \((m)\)-singular generalized measure
\[
s' : \mathcal{M} \rightarrow E^n, \mathcal{N} \rightarrow C^n
\]
and a ("essentially" unique) map
\[
f : S \rightarrow E^n, \mathcal{N} \rightarrow C^n
\]
\((\mathcal{M})\)-measurable and \((m)\)-integrable on \((S, \cdot)\) with
\[
\mu = \int f \, dm + s'
\]
(Note 3 applies here.)

**Proof**

By Theorem 2 and Note 5, \(\mu = s' + s''\) for some (unique) generalized measures \(s'\) and \(s''\) \((\mathcal{M})\)-measurable and \((m)\)-integrable on \((S, \cdot)\) with
\[
\|s''\| < \infty \quad \text{and} \quad \|s'\| < \infty
\]
(Note 3 applies here.)

Now use Theorem 1 to represent \(s'\) as \(\int f \, dm\) with \(f\) as stated. This yields the result.

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