7.2: Weighted Voting

Voting Power

There are some types of elections where the voters do not all have the same amount of power. This happens often in the business world where the power that a voter possesses may be based on how many shares of stock he/she owns. In this situation, one voter may control the equivalent of 100 votes where other voters only control 15 or 10 or fewer votes. Therefore, the amount of power that each voter possesses is different. Another example is in how the President of the United States is elected. When a person goes to the polls and casts a vote for President, he or she is actually electing who will go to the Electoral College and represent that state by casting the actual vote for President. Each state has a certain number of Electoral College votes, which is determined by the number of Senators and number of Representatives in Congress. Some states have more Electoral College votes than others, so some states have more power than others. How do we determine the power that each state possesses?

To figure out power, we need to first define some concepts of a weighted voting system. The individuals or entities that vote are called players. The notation for the players is \(P_{1}, P_{2}, P_{3}, \ldots, P_{N}\), where \(N\) is the number of players. Each player controls a certain number of votes, which are called the weight of that player. The notation for the weights is \(w_{1}, w_{2}, w_{3}, \ldots, w_{N}\), where \(w_{1}\) is the weight of \(P_{1}\), \(w_{2}\) is the weight of \(P_{2}\), etc. In order for a motion to pass, it must have a minimum number of votes. This minimum is known as the quota. The notation for quota is \(q\). The quota must be over half the total weights and cannot be more than total weight. In other words:

\[
\frac{w_{1} + w_{2} + w_{3} + \ldots + w_{N}}{2} < q \leq w_{1} + w_{2} + w_{3} + \ldots + w_{N}
\]

The way to denote a weighted voting system is \(\left[q: w_{1}, w_{2}, w_{3}, \ldots, w_{N}\right]\).
Example \((\PageIndex{1})\): Weighted Voting System

A company has 5 shareholders. Ms. Lee has 30% ownership, Ms. Miller has 25%, Mr. Matic has 22% ownership, Ms. Pierce has 14%, and Mr. Hamilton has 9%. There is a motion to decide where best to invest their savings. The company’s by-laws define the quota as 58%. What does this voting system look like?

Solution

Treating the percentages of ownership as the votes, the system looks like: \([58: 30,25,22,14,9]\)

Example \((\PageIndex{2})\): Valid Weighted Voting System

Which of the following are valid weighted voting systems?

1. \([8: 5,4,4,3,2]\)

The quota is 8 in this example. The total weight is

\[5 + 4 + 4 + 3 + 2 = 18\]

Half of 18 is 9, so the quota must be

\[9 < q \leq 18\]

Since the quota is 8, and 8 is not more than 9, this system is not valid.

2. \([16: 6,5,3,1]\)

The quota is 16 in this example. The total weight is

\[6 + 5 + 3 + 1 = 15\]

Half of 15 is 7.5, so the quota must be

\[7.5 < q \leq 15\]

Since the quota is 16, and 16 is more than 15, this system is not valid.

3. \([9: 5,4,4,3,1]\)

The quota is 9 in this example. The total weight is

\[5 + 4 + 4 + 3 + 1 = 17\]

Half of 17 is 8.5, so the quota must be

\[8.5 < q \leq 17\]

Since the quota is 9, and 9 is more than 8.5 and less than 17, this system is valid.

4. \([16: 5,4,3,3,1]\)

The quota is 16 in this example. The total weight is

\[5 + 4 + 3 + 3 + 1 = 16\]
Half of 16 is 8, so the quota must be

\[ 8 < q \leq 16 \]

Since the quota is 16, and 16 is equal to the maximum of the possible values of the quota, this system is valid. In this system, all of the players must vote in favor of a motion in order for the motion to pass.

5. \([9: 10,3,2]\)

The quota is 9 in this example. The total weight is

\[ 10 + 3 + 2 = 15 \]

Half of 15 is 7.5, so the quota must be

\[ 7.5 < q \leq 15 \]

Since the quota is 9, and 9 is between 7.5 and 15, this system is valid.

6. \([8: 5,4,2]\)

The quota is 8 in this example. The total weight is

\[ 5 + 4 + 2 = 11 \]

Half of 11 is 5.5, so the quota must be

\[ 5.5 < q \leq 11 \]

Since the quota is 8, and 8 is between 5.5 and 11, the system is valid.

In Example \(\PageIndex{2}\), some of the weighted voting systems are valid systems. Let’s examine these for some concepts. In the system

\[ [9:10,3,2] \]

player one has a weight of 10. Since the quota is nine, this player can pass any motion it wants to. So, player one holds all the power. A player with all the power that can pass any motion alone is called a **dictator**. In the system

\[ [16:5,4,3,3,1] \]

every player has the same amount of power since all players are needed to pass a motion. That also means that any player can stop a motion from passing. A player that can stop a motion from passing is said to have **veto power**. In the system

\[ [8:5,4,2] \]

player three has a weight of two. Players one and two can join together and pass any motion without player three, and player three doesn’t have enough weight to join with either player one or player two to pass a motion. So player three has no power. A player who has no power is called a **dummy**.

Example \(\PageIndex{3}\): Dictator, Veto Power, or Dummy?

In the weighted voting system \([57: 23,21,16,12]\), are any of the players a dictator or a dummy or do any have veto power.

**Solution**
Since no player has a weight higher than or the same as the quota, then there is no dictator. If players one and two join together, they can’t pass a motion without player three, so player three has veto power. Under the same logic, players one and two also have veto power. Player four cannot join with any players to pass a motion, so player four’s votes do not matter. Thus, player four is a dummy.

Now that we have an understanding of some of the basic concepts, how do we quantify how much power each player has? There are two different methods. One is called the Banzhaf Power Index and the other is the Shapely-Shubik Power Index. We will look at each of these indices separately.

**Banzhaf Power Index**

A coalition is a set of players that join forces to vote together. If there are three players \( \{P_1\}, \{P_2\}, \{P_3\} \) then the coalitions would be: \( \{P_1\}, \{P_2\}, \{P_3\}, \{P_1, P_2\}, \{P_1, P_3\}, \{P_2, P_3\}, \{P_1, P_2, P_3\} \).

Not all of these coalitions are winning coalitions. To find out if a coalition is winning or not look at the sum of the weights in each coalition and then compare that sum to the quota. If the sum is the quota or more, then the coalition is a winning coalition.

**Example**: Coalitions with Weights

In the weighted voting system \( [17: 12,7,3] \), the weight of each coalition and whether it wins or loses is in the table below.

<table>
<thead>
<tr>
<th>Coalition</th>
<th>Weight</th>
<th>Win or Lose?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( {P_1} )</td>
<td>12</td>
<td>Lose</td>
</tr>
<tr>
<td>( {P_2} )</td>
<td>7</td>
<td>Lose</td>
</tr>
<tr>
<td>( {P_3} )</td>
<td>3</td>
<td>Lose</td>
</tr>
<tr>
<td>( {P_1, P_2} )</td>
<td>19</td>
<td>Win</td>
</tr>
<tr>
<td>( {P_1, P_3} )</td>
<td>15</td>
<td>Lose</td>
</tr>
</tbody>
</table>
In each of the winning coalitions you will notice that there may be a player or players that if they were to leave the coalition, the coalition would become a losing coalition. If there is such a player or players, they are known as the critical player(s) in that coalition.

Example (PageIndex{5})\): Critical Players

In the weighted voting system (\([17: 12,7,3]\)), determine which player(s) are critical player(s). Note that we have already determined which coalitions are winning coalitions for this weighted voting system in Example (PageIndex{4})\). Thus, when we continue on to determine the critical player(s), we only need to list the winning coalitions.

<table>
<thead>
<tr>
<th>Coalition</th>
<th>Weight</th>
<th>Win or Lose?</th>
<th>Critical Player</th>
</tr>
</thead>
<tbody>
<tr>
<td>([P_2, P_3])</td>
<td>10</td>
<td>Lose</td>
<td></td>
</tr>
<tr>
<td>([P_1, P_2, P_3])</td>
<td>22</td>
<td>Win</td>
<td></td>
</tr>
</tbody>
</table>

Notice, player one and player two are both critical players two times and player three is never a critical player.

**Banzhaf Power Index**

The Banzhaf power index is one measure of the power of the players in a weighted voting system. In this index, a player's power is determined by the ratio of the number of times that player is critical to the total number of times any and all players are critical.

**Definition: Banzhaf Power Index**

**Banzhaf Power Index** for Player

\([p_i=\dfrac{B_i}{T}]\)
where \(B_i\) is number of times player \(P_i\) is critical and \(T\) is total number of times all players are critical.

Example \(\PageIndex{6}\): Banzhaf Power Index

In the weighted voting system \([17: 12,7,3]\), determine the Banzhaf power index for each player.

Solution

Using Table \(\PageIndex{2}\), Player one is critical two times, Player two is critical two times, and Player three is never critical. So \(T = 4, B1 = 2, B2 = 2,\) and \(B3 = 0\). Thus:

- Banzhaf power index of \(P1\) is
  \[
  \frac{2}{4} = \frac{1}{2} = 0.5 = 50\%
  \]
- Banzhaf power index of \(P2\) is
  \[
  \frac{2}{4} = \frac{1}{2} = 0.5 = 50\%
  \]
- Banzhaf power index of \(P3\) is
  \[
  \frac{0}{4} = 0
  \]

Thus:

- Banzhaf power index of \(P1\) is 50%.
- Banzhaf power index of \(P2\) is 50%.
- Banzhaf power index of \(P3\) is 0%.

So players one and two each have 50% of the power. This means that they have equal power, even though player one has five more votes than player two. Also, player three has 0% of the power and so player three is a dummy.

How many coalitions are there? From the last few examples, we know that if there are three players in a weighted voting system, then there are seven possible coalitions. How about when there are four players?

Table \(\PageIndex{3}\): Coalitions with Four Players

<table>
<thead>
<tr>
<th>1 Player</th>
<th>2 Players</th>
<th>3 Players</th>
</tr>
</thead>
<tbody>
<tr>
<td>({P_1})</td>
<td>({P_1, P_2}), ({P_1, P_3}), ({P_1, P_4})</td>
<td>({P_1, P_2, P_3}), ({P_1, P_2, P_4}), ({P_1, P_3, P_4})</td>
</tr>
<tr>
<td>({P_2})</td>
<td>({P_2, P_3}), ({P_2, P_4})</td>
<td>({P_2, P_3, P_4})</td>
</tr>
<tr>
<td>({P_3})</td>
<td>({P_3, P_4})</td>
<td>({P_3, P_4})</td>
</tr>
<tr>
<td>({P_4})</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

So when there are four players, it turns out that there are 15 coalitions. When there are five players, there are 31 coalitions (there are too many to list, so take my word for it). It doesn’t look like there is a pattern to the number of coalitions, until you realize that 7, 15, and 31 are all one less than a power of two. In fact, seven is one less than 8, which is \(2^3\), and so on.
2^3, 15 is one less than
2^4, and 31 is one less than
2^5. So it appears that the number of coalitions for N players is
2^N - 1.

Example \(\PageIndex{7}\): Banzhaf Power Index

Example \(\PageIndex{1}\) had the weighted voting system of \([58: 30,25,22,14,9]\). Find the Banzhaf power index for each player.

**Solution**

Since there are five players, there are 31 coalitions. This is too many to write out, but if we are careful, we can just write out the winning coalitions. No player can win alone, so we can ignore all of the coalitions with one player. Also, no two-player coalition can win either. So we can start with the three player coalitions.

**Table \(\PageIndex{4}\): Winning Coalitions and Critical Players**

<table>
<thead>
<tr>
<th>Winning Coalition</th>
<th>Critical Player</th>
</tr>
</thead>
<tbody>
<tr>
<td>({P_1, P_2, P_3})</td>
<td>(P_1, P_2, P_3)</td>
</tr>
<tr>
<td>({P_1, P_2, P_4})</td>
<td>(P_1, P_2, P_4)</td>
</tr>
<tr>
<td>({P_1, P_2, P_5})</td>
<td>(P_1, P_2, P_5)</td>
</tr>
<tr>
<td>({P_1, P_3, P_4})</td>
<td>(P_1, P_3, P_4)</td>
</tr>
<tr>
<td>({P_1, P_3, P_5})</td>
<td>(P_1, P_3, P_5)</td>
</tr>
</tbody>
</table>
So player one is critical eight times, player two is critical six times, player three is critical six times, player four is critical four times, and player five is critical two times. Thus, the total number of times any player is critical is $T = 26$.

- Banzhaf power index for $P1 =$
  \[
  \frac{8}{26} = \frac{4}{13} = 0.308 = 30.8\%
  \]
- Banzhaf power index for $P2 =$
  \[
  \frac{6}{26} = \frac{3}{13} = 0.231 = 23.1\%
  \]
- Banzhaf power index for $P3 =$
  \[
  \frac{6}{26} = \frac{3}{13}
  \]

Winning Coalition  Critical Player

\[
\begin{align*}
\{P_2, P_3, P_4\} & \quad P_2, P_3, P_4 \\
\{P_1, P_2, P_3, P_4\} & \\
\{P_1, P_2, P_3, P_5\} & \quad P_1 \\
\{P_1, P_2, P_4, P_5\} & \quad P_1, P_2 \\
\{P_1, P_3, P_4, P_5\} & \quad P_1, P_3 \\
\{P_2, P_3, P_4, P_5\} & \quad P_2, P_3, P_4 \\
\{P_1, P_2, P_3, P_4, P_5\} & 
\end{align*}
\]
Banzhaf power index for $P_4 = \frac{4}{26} = \frac{2}{13} = 0.154 = 15.4$

Banzhaf power index for $P_5 = \frac{2}{26} = \frac{1}{13} = 0.077 = 7.7$

Every player has some power. Player one has the most power with 30.8% of the power. No one has veto power, since no player is in every winning coalition.

Shapely-Shubik Power Index

Shapely-Shubik takes a different approach to calculating the power. Instead of just looking at which players can form coalitions, Shapely-Shubik decided that all players form a coalition together, but the order that players join a coalition is important. This is called a sequential coalition. Instead of looking at a player leaving a coalition, this method examines what happens when a player joins a coalition. If when a player joins the coalition, the coalition changes from a losing to a winning coalition, then that player is known as a pivotal player. Now we count up how many times each player is pivotal, and then divide by the number of sequential coalitions. Note, that in reality when coalitions are formed for passing a motion, not all players will join the coalition. The sequential coalition is used only to figure out the power each player possess.

As an example, suppose you have the weighted voting system of

$[17:12,7,3]$.

One of the sequential coalitions is

$\langle P_1, P_2, P_3 \rangle$

which means that $P_1$ joins the coalition first, followed by $P_2$ joining the coalition, and finally, $P_3$ joins the coalition. When player one joins the coalition, the coalition is a losing coalition with only 12 votes. Then, when player two joins, the coalition now has enough votes to win (12 + 7 = 19 votes). Player three joining doesn’t change the coalition’s winning status so it is irrelevant. Thus, player two is the pivotal player for this coalition. Another sequential coalition is

$\langle P_1, P_2, P_3 \rangle$

. When player one joins the coalition, the coalition is a losing coalition with only 12 votes. Then player three joins but the coalition is still a losing coalition with only 15 votes. Then player two joins and the coalition is now a winning coalition with 22 votes. So player two is the pivotal player for this coalition as well.

How many sequential coalitions are there for $N$ players? Let’s look at three players first. The sequential coalitions for three players ($P_1, P_2, P_3$) are:
\[
\{P_1, P_2, P_3\}, \{P_1, P_2, P_4\}, \{P_1, P_3, P_4\}, \{P_2, P_3, P_4\}, \{P_1, P_2, P_1\}, \{P_3, P_4, P_1\}
\]

Note: The difference in notation: We use

\[
\{ \}
\]

for coalitions and

\[
(\}
\]

sequential coalitions.

So there are six sequential coalitions for three players. Can we come up with a mathematical formula for the number of sequential coalitions? For the first player in the sequential coalition, there are 3 players to choose from. Once you choose one for the first spot, then there are only 2 players to choose from for the second spot. The third spot will only have one player to put in that spot. Notice, \(3 \times 2 \times 1 = 6\). It looks like if you have \(N\) players, then you can find the number of sequential coalitions by multiplying

\[
N (N - 1)(N - 2) \cdots (3)(2)(1)
\]

. This expression is called a \(N\) factorial, and is denoted by \(N!\).

Most calculators have a factorial button. The process for finding a factorial on the TI-83/84 is demonstrated in the following example.

Example \(\PageIndex{8}\): Finding a Factorial on the TI-83/84 Calculator

Find 5! on the TI-83/84 Calculator.

First, note that

\[5! = 5 \times 4 \times 3 \times 2 \times 1\]

, which is easy to do without the special button on the calculator, but we will use it anyway. First, input the number five on the home screen of the calculator.

![Figure](5 Entered on the Home Screen)

Then press the MATH button. You will see the following:
Now press the right arrow key to move over to the abbreviation PRB, which stands for probability.

Number 4:! is the factorial button. Either arrow down to the number four and press ENTER, or just press the four button. This will put the ! next to your five on the home screen.

Now press ENTER and you will see the result.
Notice that 5! is a very large number. So if you have 5 players in the weighted voting system, you will need to list 120 sequential coalitions. This is quite large, so most calculations using the Shapely-Shubik power index are done with a computer.

Now we have the concepts for calculating the Shapely-Shubik power index.

**Shapely-Shubik Power Index** for Player \( P_i \)

\[
\text{Shapely-Shubik Power Index} = \frac{S_i}{N!}
\]

where

\( S_i \)

is how often the player is pivotal

\( N \) is the number of players and \( N! \) is the number of sequential coalitions

Example \( \PageIndex{9} \): Shapely-Shubik Power Index

In the weighted voting system \( \{17: 12, 7, 3\} \), determine the Shapely-Shubik power index for each player.

**Solution**

First list every sequential coalition. Then determine which player is pivotal in each sequential coalition. There are 3! = 6 sequential coalitions.
Table 1: Sequential Coalitions and Pivotal Players

<table>
<thead>
<tr>
<th>Sequential coalition</th>
<th>Pivotal player</th>
</tr>
</thead>
<tbody>
<tr>
<td>( {P_1, P_2, P_3} )</td>
<td>( P_2 )</td>
</tr>
<tr>
<td>( {P_1, P_2, P_3} )</td>
<td>( P_2 )</td>
</tr>
<tr>
<td>( {P_2, P_3, P_1} )</td>
<td>( P_1 )</td>
</tr>
<tr>
<td>( {P_2, P_3, P_1} )</td>
<td>( P_1 )</td>
</tr>
<tr>
<td>( {P_1, P_2, P_3} )</td>
<td>( P_2 )</td>
</tr>
<tr>
<td>( {P_3, P_2, P_1} )</td>
<td>( P_1 )</td>
</tr>
</tbody>
</table>

So,
\[ S_1 = 3 \]
\[ S_2 = 3 \]
, and
\[ S_3 = 0 \]

Shapely-Shubik power index for \( P_1 \)
\[ \frac{3}{6} - \frac{1}{2} = 0.5 = 50\% \]

Shapely-Shubik power index for \( P_2 \)
\[
\frac{3}{6} = \frac{1}{2}
\]

= 0.5 = 50%

Shapely-Shubik power index for \( P3 \)

\[
\frac{0}{6} = 0
\]

= 0%

This is the same answer as the Banzhaf power index. The two methods will not usually produce the same exact answer, but their answers will be close to the same value. Notice that player three is a dummy using both indices.

Example \( \PageIndex{10} \): Calculating the Power

For the voting system

\[ [7 : 6, 4, 2] \]

, find:

1. The Banzhaf power index for each player

The first thing to do is list all of the coalitions and determine which ones are winning and which ones are losing. Then determine the critical player(s) in each winning coalition.

Table \( \PageIndex{11} \) Coalitions and Critical Players

<table>
<thead>
<tr>
<th>Coalition</th>
<th>Weight</th>
<th>Win or Lose?</th>
<th>Critical Player</th>
</tr>
</thead>
<tbody>
<tr>
<td>{ P_1 }</td>
<td>6</td>
<td>Lose</td>
<td></td>
</tr>
<tr>
<td>{ P_2 }</td>
<td>4</td>
<td>Lose</td>
<td></td>
</tr>
<tr>
<td>{ P_3 }</td>
<td>2</td>
<td>Lose</td>
<td></td>
</tr>
<tr>
<td>{ P_1, P_2 }</td>
<td>10</td>
<td>Win</td>
<td>P1, P2</td>
</tr>
</tbody>
</table>
\[
P_1, P_3 \quad 8 \quad \text{Win} \quad P1, P3
\]

\[
P_2, P_3 \quad 6 \quad \text{Lose}
\]

\[
P_1, P_2, P_3 \quad 12 \quad \text{Win} \quad P1
\]

So,

\[B_1 = 3, \ B_2 = 1, \ B_3 = 1, \ T = 3 + 1 + 1 = 5\]

Banzhaf power index of \(P1\)

\[
\frac{3}{5} = 0.6 = 60\%
\]

Banzhaf power index of \(P2\)

\[
\frac{1}{5} = 0.2 = 20\%
\]

Banzhaf power index of \(P3\)

\[
\frac{1}{5} = 0.2 = 20\%
\]

2. The Shapely-Shubik power index for each player

The first thing to do is list all of the sequential coalitions, and then determine the pivotal player in each sequential coalition.

<table>
<thead>
<tr>
<th>Sequential Coalition</th>
<th>Pivotal Player</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P_1, P_2, P_3)</td>
<td>(P2)</td>
</tr>
</tbody>
</table>
Sequential Coalition  Pivotal Player

\( \{P_1, P_2, P_3\} \quad P_3 \)

\( \{P_2, P_1, P_3\} \quad P_1 \)

\( \{P_2, P_3, P_1\} \quad P_1 \)

\( \{P_3, P_2, P_1\} \quad P_1 \)

\( \{P_1, P_3, P_2\} \quad P_1 \)

So

\[ S_1 = 4, \quad S_2 = 1, \quad S_3 = 1, \quad 3! = 6 \]

Shapely-Shubik power index of \( P_1 \)

\[
\frac{4}{6} = \frac{2}{3}
\]

\( = 0.667 = 66.7\% \)

Shapely-Shubik power index of \( P_2 \)

\[
\frac{1}{6}
\]

\( = 0.167 = 16.7\% \)

Shapely-Shubik power index of \( P_3 \)

\[
\frac{1}{6}
\]

\( = 0.167 = 16.7\% \)

Notice the two indices give slightly different results for the power distribution, but they are close to the same values.