3.2.1: Quasilinear Elliptic Equations

There is a large class of quasilinear equations such that the associated characteristic equation has no solution \(\chi, \nabla \chi \not= 0\).

Set

\[
U = \{(x,z,p) : x \in \Omega, z \in \mathbb{R}^1, p \in \mathbb{R}\}.
\]

**Definition.** The quasilinear equation (3.2.1) is called *elliptic* if the matrix \((a^{ij}(x,z,p))\) is positive definite for each \((x,z,p) \in U\).

Assume equation (3.2.1) is elliptic and let \(\lambda(x,z,p)\) be the minimum and \(\Lambda(x,z,p)\) the maximum of the eigenvalues of \((a^{ij}(x,z,p))\), then

\[
0 < \lambda(x,z,p)|\zeta|^2 \leq \sum_{i,j=1}^n a^{ij}(x,z,p)\zeta_i\zeta_j \leq \Lambda(x,z,p)|\zeta|^2
\]

for all \(\zeta \in \mathbb{R}\).

**Definition.** Equation (3.2.1) is called *uniformly elliptic* if \((\Lambda(x,z,p))/\lambda(x,z,p)\) is uniformly bounded in \(U\).

An important class of elliptic equations which are not uniformly elliptic (non-uniformly elliptic) is

\[
\begin{align*}
\text{\begin{equation} \tag{3.2.1.1} \nonumber
\sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \frac{u_{x_i}}{\sqrt{1+|\nabla u|^2}} \right) + \text{lower order terms} = 0.
\end{equation}\end{align*}
\]
The main part is the minimal surface operator (left hand side of the minimal surface equation). The coefficients \(a^{ij}\) are

\[
a^{ij}(x,z,p) = \left(1+|p|^2\right)^{-1/2}\left(\delta_{ij} - \frac{p_ip_j}{1+|p|^2}\right),
\]

\(\delta_{ij}\) denotes the Kronecker delta symbol. It follows that

\[
\lambda = \frac{1}{\left(1+|p|^2\right)^{3/2}}, \quad \Lambda = \frac{1}{\left(1+|p|^2\right)^{1/2}}.
\]

Thus equation (ref{nonuniform}) is not uniformly elliptic.

The behavior of solutions of uniformly elliptic equations is similar to linear elliptic equations in contrast to the behavior of solutions of non-uniformly elliptic equations. Typical examples for non-uniformly elliptic equations are the minimal surface equation and the capillary equation.

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