Appendix E: Summary of Notation Used

In addition to the notation for sets and functions (as reviewed in Appendix B), the notation for matrices and linear systems, and the common mathematical symbols reviewed in Appendix D, the following notation is used frequently in the study of Linear Algebra.

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Special Sets

1. The set of **positive integers** is denoted by \( \mathbb{Z}_{+} = \{1, 2, 3, 4, \ldots\} \).
2. The set of **integers** is denoted by \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \).
3. The set of **real numbers** is denoted by \( \mathbb{R} \).
4. The set of **complex numbers** is denoted by \( \mathbb{C} = \{ x + y i \ | \ x, y \in \mathbb{R} \} \). (\( \mathbb{F} \) is often used to denote a set that can equally well be chosen as either \( \mathbb{R} \) or \( \mathbb{C} \).)
5. The set of **polynomials of degree at most** \( n \) in the variable \( z \) and with coefficients over \( \mathbb{F} \) is denoted by \( \mathbb{F}[z] = \{ a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \ | \ a_0, a_1, \ldots, a_n \in \mathbb{F} \} \).
6. The set of **polynomials** of all degrees in \( z \) with coefficients over \( \mathbb{F} \) is denoted by \( \mathbb{F}[z] \).
7. The set of **matrices** of size \( m \times n \) over \( \mathbb{F} \) is denoted by \( \mathbb{F}^{m \times n} \).
8. The **general linear group** of \( n \times n \) invertible matrices over \( \mathbb{F} \) is denoted by \( GL(n, \mathbb{F}) \).
9. The set of **continuous functions** with domain \( \mathbb{R} \) and codomain \( \mathbb{R} \) is denoted by \( \mathcal{C}(\mathbb{R}) \), and the set of **smooth** (a.k.a. **infinitely differentiable**) functions with domain \( \mathbb{R} \) and codomain \( \mathbb{R} \) is denoted by \( \mathcal{C}^\infty(\mathbb{R}) \).
Complex Numbers

Given \( z = x + y i \in \mathbb{C} \) with \( x, y \in \mathbb{R} \), and where \( i \) denotes the imaginary unit, we denote

1. the **additive inverse** of \( z \) by \( \mathrm{-}z = (\mathrm{-}x) + (\mathrm{-}y)i \).
2. the **multiplicative inverse** of \( z \) by \( z^{-1} = \left(\frac{x}{x^2 + y^2}\right) + \left(\frac{-y}{x^2 + y^2}\right)i \), assuming \( z \neq 0 \).
3. the **complex conjugate** of \( z \) by \( \overline{z} = x + (\mathrm{-}y)i \).
4. the **real part** of \( z \) by \( \RealPart(z) = x \).
5. the **imaginary part** of \( z \) by \( \ImaginaryPart(z) = y \).
6. the **modulus** of \( z \) by \( |z| = \sqrt{x^2 + y^2} \).
7. the **argument** of \( z \) by \( \Argument(z) = \min_{\theta \geq 0} \{ \theta \mid x = \cos(\theta), y = \sin(\theta) \} \).

Vector Spaces

Let \( \mathcal{V} \) be an arbitrary vector space, and let \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) be subspaces of \( \mathcal{V} \). Then we denote

1. the **additive identity** of \( \mathcal{V} \) by \( 0 \).
2. the **additive inverse** of each \( v \in \mathcal{V} \) by \( \mathrm{-}v \).
3. the **(subspace) sum** of \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) by \( \mathcal{U}_1 + \mathcal{U}_2 \).
4. the **direct sum** of \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) by \( \mathcal{U}_1 \oplus \mathcal{U}_2 \).
5. the **span** of \( v_1, v_2, \ldots, v_n \in \mathcal{V} \) by \( \Span(v_1, v_2, \ldots, v_n) \).
6. the **dimension** of \( \mathcal{V} \) by \( \dim(\mathcal{V}) \), where

\[
\dim(\mathcal{V}) = \begin{cases} 0 & \text{if } \mathcal{V} = \{0\} \text{ is the zero vector space}, \\ n & \text{if every basis for } \mathcal{V} \text{ has } n \text{ elements in it}, \\ \infty & \text{otherwise}. \end{cases}
\]
7. the **change of basis map** with respect to a given basis \( \mathcal{B} \) for \( \mathcal{V} \) by \( \left[ \cdot \right]_\mathcal{B} : \mathcal{V} \to \mathbb{R}^n \), where \( \mathcal{V} \) is assumed to be \( (n) \)-dimensional.

Linear Maps

Let \( \mathcal{U}, \mathcal{V}, \) and \( \mathcal{W} \) denote vector spaces over the field \( \mathbb{F} \). Then we denote

1. the vector space of all linear maps from \( \mathcal{V} \) into \( \mathcal{W} \) by \( \mathcal{L}(\mathcal{V}, \mathcal{W}) \) or \( \mathrm{Hom}_{\mathbb{F}}(\mathcal{V}, \mathcal{W}) \).
2. the vector space of all linear operators on \( \mathcal{V} \) by \( \mathcal{L}(\mathcal{V}) \) or \( \mathrm{Hom}_{\mathbb{F}}(\mathcal{V}) \).
3. the **composition** (a.k.a. product) of \( S \in \mathcal{L}(\mathcal{U}, \mathcal{V}) \) and \( T \in \mathcal{L}(\mathcal{V}, \mathcal{W}) \) by \( T \circ S \) (or, equivalently, \( TS \)), where \( (T \circ S)(u) = T(S(u)) \) for each \( u \in \mathcal{U} \).
4. the **null space** (a.k.a. kernel) of \( T \in \mathcal{L}(\mathcal{V}, \mathcal{W}) \) by \( \kernel(T) = \{ v \in \mathcal{V} \mid T(v) = 0 \} \).
5. the **range** of \( T \in \mathcal{L}(\mathcal{V}, \mathcal{W}) \) by \( \range(T) = \{ w \in \mathcal{W} \mid \exists v \in \mathcal{V} \mid T(v) = w \} \).
6. the **eigenspace** of \( T \in \mathcal{L}(\mathcal{V}) \) associated to eigenvalue \( \lambda \in \mathbb{C} \) by \( V_{\lambda} = \kernel(T - \lambda \mathrm{id}_{\mathcal{V}}) \), where \( \mathrm{id}_{\mathcal{V}} \) denotes the identity map on \( \mathcal{V} \).
7. the matrix of \( (T \in \mathcal{L}(V, W)) \) with respect to the basis \( (B) \) on \( (V) \) and with respect to the basis \( (C) \) on \( (W) \) by \( \mathcal{M}(T, B, C) \) (or simply as \( \mathcal{M}(T) \)).

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**Inner Product Spaces**

Let \( (V) \) be an arbitrary inner product space, and let \( (U) \) be a subspace of \( (V) \). Then we denote

1. the **inner product** on \( (V) \) by \( \langle \cdot, \cdot \rangle \).
2. the **norm** on \( (V) \) induced by \( \langle \cdot, \cdot \rangle \) as \( \| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle} \).
3. the **orthogonal complement** of \( (U) \) by \( (U^\perp) = \{ v \in V \mid \langle u, v \rangle = 0, \forall u \in U \} \).
4. the **orthogonal projection** onto \( (U) \) by \( (P_{(U)}) \), which, for each \( (v \in V) \), is defined by \( (P_{(U)}(v) = u) \) such that \( (v = u + w) \) for \( (u \in U) \) and \( (w \in U^\perp) \).
5. the **adjoint** of the operator \( (T \in \mathcal{L}(V)) \) by \( (T^*) \), where \( (T^*) \) satisfies \( \langle T(v), w \rangle = \langle v, T^*(w) \rangle \) for each \( (v, w \in V) \).
6. the **square root** of the positive operator \( (T \in \mathcal{L}(V)) \) by \( (\sqrt{T}) \), which satisfies \( (T = \sqrt{T} \sqrt{T}) \).
7. the **positive part** of the operator \( (T \in \mathcal{L}(V)) \) by \( (|T| = \sqrt{T^* T}) \).

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