Intersection of a Line and a Plane

A given line and a given plane may or may not intersect. If the line does intersect with the plane, it's possible that the line is completely contained in the plane as well. How can we differentiate between these three possibilities?

Example (PageIndex{8}): Finding the intersection of a Line and a plane

Determine whether the following line intersects with the given plane. If they do intersect, determine whether the line is contained in the plane or intersects it in a single point. Finally, if the line intersects the plane in a single point, determine this point of intersection.

\[
\begin{align*}
\text{Line:} & \quad x &= 2 - t & \text{Plane:} & \quad 3x - 2y + z &= 10 \\
& \quad y &= 1 + t & & \\
& \quad z &= 3t & & \\
\end{align*}
\]

Solution

Notice that we can substitute the expressions of \(t\) given in the parametric equations of the line into the plane equation for \(x\), \(y\), and \(z\).

\[
\begin{align*}
3(2-t) - 2(1+t) + 3t &= 10 \\
6 - 3t - 2 - 2t + 3t &= 10 \\
4 - 2t &= 10 \\
\end{align*}
\]
\[-2t = 6\]
\[t = -3\]

Since we found a single value of \(t\) from this process, we know that the line should intersect the plane in a single point, here where \(t = -3\). So the point of intersection can be determined by plugging this value in for \(t\) in the parametric equations of the line.

Here: \(x = 2 - (-3) = 5, \quad y = 1 + (-3) = -2, \quad z = 3(-3) = -9\).

So the point of intersection of this line with this plane is \((5, -2, -9)\). We can verify this by putting the coordinates of this point into the plane equation and checking to see that it is satisfied.

Check: \(3(5) - 2(-2) + (-9) = 15 + 4 - 9 = 10\) \(\checkmark\)

Now that we have examined what happens when there is a single point of intersection between a line and a point, let's consider how we know if the line either does not intersect the plane at all or if it lies on the plane (i.e., every point on the line is also on the plane).

Example \(\PageIndex{9}\): Other relationships between a line and a plane

Determine whether the following line intersects with the given plane. If they do intersect, determine whether the line is contained in the plane or intersects it in a single point. Finally, if the line intersects the plane in a single point, determine this point of intersection.

\[
\begin{align*}
\text{Line:} & \quad x &= 1 + 2t & \text{Plane:} & \quad x + 2y - 2z = 5 \\
& \quad y &= -2 + 3t \\
& \quad z &= -1 + 4t
\end{align*}
\]

Solution

Substituting the expressions of \((t)\) given in the parametric equations of the line into the plane equation gives us:

\[
[(1+2t) +2(-2+3t) - 2(-1 + 4t) = 5\]

Simplifying the left side gives us:

\[
[1 + 2t -4 + 6t + 2 - 8t = 5\]

Collecting like terms on the left side causes the variable \((t)\) to cancel out and leaves us with a contradiction:

\[-1 = 5\]

Since this is not true, we know that there is no value of \((t)\) that makes this equation true, and thus there is no value of \((t)\) that will give us a point on the line that is also on the plane. This means that this line does not intersect with this plane and there will be no point of intersection.
How can we tell if a line is contained in the plane?

What if we keep the same line, but modify the plane equation to be \( (x + 2y - 2z = -1) \)? In this case, repeating the steps above would again cause the variable \( t \) to be eliminated from the equation, but it would leave us with an identity, \((-1 = -1)\), rather than a contradiction. This means that every value of \( t \) will produce a point on the line that is also on the plane, telling us that the line is contained in the plane whose equation is \( (x + 2y - 2z = -1) \).