14.2: Countable and Uncountable Sets

Let’s summarize the main points from the previous section.

1. \(|A| = |B|\) if and only if there exists a bijection \(A \rightarrow B\).
2. \(|\mathbb{N}| = |\mathbb{Z}|\) because there exists a bijection \(\mathbb{N} \rightarrow \mathbb{Z}\).
3. \(|\mathbb{N}| \neq |\mathbb{R}|\) because there exists no bijection \(\mathbb{N} \rightarrow \mathbb{R}\).

Thus, even though \(|\mathbb{N}|, |\mathbb{Z}|\) and \(|\mathbb{R}|\) are all infinite sets, their cardinalities are not all the same. The sets \(\mathbb{N}\) and \(\mathbb{Z}\) have the same cardinality, but \(|\mathbb{R}|\)’s cardinality is different from that of both the other sets. This means infinite sets can have different sizes. We now make some definitions to put words and symbols to this phenomenon.

In a certain sense you can count the elements of \(|\mathbb{N}|\); you can count its elements off as 1, 2, 3, 4, ..., but you’d have to continue this process forever to count the whole set. Thus we will call \(|\mathbb{N}|\) a countably infinite set, and the same term is used for any set whose cardinality equals that of \(|\mathbb{N}|\).

Definition 14.2

Suppose \(A\) is a set. Then \(A\) is **countably infinite** if \(|\mathbb{N}| = |A|\), that is, if there exists a bijection \(\mathbb{N} \rightarrow A\). The set \(A\) is **countable** if it is finite or countably infinite. The set \(A\) is **uncountable** if it is infinite and \(|\mathbb{N}| \neq |A|\), that is, if \(A\) is infinite and there is no bijection \(\mathbb{N} \rightarrow A\).

Thus \(|\mathbb{Z}|\) is countably infinite but \(|\mathbb{R}|\) is uncountable. This section deals mainly with countably infinite sets. Uncountable sets are treated later.

If \(A\) is countably infinite, then \(|\mathbb{N}| = |A|\), so there is a bijection \(f : \mathbb{N} \rightarrow A\). Think of \(f\) as “counting” the elements of \(A\). The first element of \(A\) is \(f(1)\), followed by \(f(2)\), then \(f(3)\) and so on. It makes sense to
think of a countably infinite set as the smallest type of infinite set, because if the counting process stopped, the set would be finite, not infinite; a countably infinite set has the fewest elements that a set can have and still be infinite. We reserve the special symbol \(\aleph_0\) to stand for the cardinality of countably infinite sets.

**Definition 14.3**

The cardinality of the natural numbers is denoted as \(\aleph_0\). That is, \(|\mathbb{N}| = \aleph_0\). Thus any countably infinite set has cardinality \(\aleph_0\).

(The symbol \(\aleph\) is the first letter in the Hebrew alphabet, and is pronounced “aleph.” The symbol \(\aleph_0\) is pronounced “aleph naught.”) The summary of facts at the beginning of this section shows \(|\mathbb{Z}| = \aleph_0\) and \(|\mathbb{R}| \neq \aleph_0\).

**Example 14.5**

Let \(E = \{2k : k \in \mathbb{Z}\}\) be the set of even integers. The function \(f : \mathbb{Z} \rightarrow E\) defined as \(f(n) = 2n\) is easily seen to be a bijection, so we have \(|\mathbb{Z}| = |E|\). Thus, as \(|\mathbb{N}| = |\mathbb{Z}| = |E|\), the set \(E\) is countably infinite and \(|E| = \aleph_0\).

Here is a significant fact: The elements of any countably infinite set \(A\) can be written in an infinitely long list \(a_1, a_2, a_3, a_4, \ldots\) that begins with some element \(a_1 \in A\) and includes every element of \(A\). For example, the set \(E\) in the above example can be written in list form as \(0, 2, -2, 4, -4, 6, -6, 8, -8, \ldots\) The reason that this can be done is as follows. Since \(A\) is countably infinite, Definition 14.2 says there is a bijection \(f : \mathbb{N} \rightarrow A\). This allows us to list out the set \(A\) as an infinite list \(f(1), f(2), f(3), f(4), \ldots\) Conversely, if the elements of \(A\) can be written in list form as \(a_1, a_2, a_3, a_4, \ldots\), then the function \(f : \mathbb{N} \rightarrow A\) defined as \(f(n) = a_n\) is a bijection, so \(A\) is countably infinite. We summarize this as follows.

**Theorem 14.3**

A set \(A\) is countably infinite if and only if its elements can be arranged in an infinite list \(a_1, a_2, a_3, a_4, \ldots\).

As an example of how this theorem might be used, let \(P\) denote the set of all prime numbers. Since we can list its elements as \(2, 3, 5, 7, 11, 13, \ldots\), it follows that the set \(P\) is countably infinite.

As another consequence of Theorem 14.3, note that we can interpret the fact that the set \(|\mathbb{R}|\) is not countably infinite as meaning that it is impossible to write out all the elements of \(|\mathbb{R}|\) in an infinite list. (After all, we tried to do that in the table on page 271, and failed!)

This raises a question. Is it also impossible to write out all the elements of \(|\mathbb{Q}|\) in an infinite list? In other words, is the set \(|\mathbb{Q}|\) of rational numbers countably infinite or uncountable? If you start plotting the rational numbers on the number line, they seem to mostly fill up \(|\mathbb{R}|\). Sure, some numbers such as \(\sqrt{2}\), \(\pi\) and \(e\) will not be plotted, but the dots representing rational numbers seem to predominate. We might thus expect \(|\mathbb{Q}|\) to be
uncountable. However, it is a surprising fact that $\mathbb{Q}$ is countable. The proof presented below arranges all the rational numbers in an infinitely long list.

Theorem 14.4

The set $\mathbb{Q}$ of rational numbers is countably infinite.

Proof. To prove this, we just need to show how to write the set $\mathbb{Q}$ in list form. Begin by arranging all rational numbers in an infinite array. This is done by making the following chart. The top row has a list of all integers, beginning with 0, then alternating signs as they increase. Each column headed by an integer $k$ contains all the fractions (in reduced form) with numerator $k$. For example, the column headed by 2 contains the fractions $\frac{2}{1}, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \cdots$, and so on. It does not contain $\frac{2}{2}, \frac{2}{4}, \frac{2}{6}$, etc., because those are not reduced, and in fact their reduced forms appear in the column headed by 1. You should examine this table and convince yourself that it contains all rational numbers in $\mathbb{Q}$.

\[
\begin{array}{cccccccccc}
0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & -4 & 5 & -5 & \cdots \\
\hline
0 & 1 & \frac{-1}{2} & \frac{2}{3} & \frac{-3}{4} & \frac{4}{5} & \frac{-5}{6} & \frac{6}{7} & \frac{-7}{8} & \frac{8}{9} & \frac{-9}{10} & \cdots \\
\frac{1}{2} & \frac{-1}{3} & \frac{2}{4} & \frac{-3}{5} & \frac{4}{6} & \frac{-5}{7} & \frac{6}{8} & \frac{-7}{9} & \frac{8}{10} & \frac{-9}{11} & \cdots \\
\frac{1}{3} & \frac{-1}{4} & \frac{2}{5} & \frac{-3}{6} & \frac{4}{7} & \frac{-5}{8} & \frac{6}{9} & \frac{-7}{10} & \frac{8}{11} & \frac{-9}{12} & \cdots \\
\frac{1}{4} & \frac{-1}{5} & \frac{2}{6} & \frac{-3}{7} & \frac{4}{8} & \frac{-5}{9} & \frac{6}{10} & \frac{-7}{11} & \frac{8}{12} & \frac{-9}{13} & \cdots \\
\frac{1}{5} & \frac{-1}{6} & \frac{2}{7} & \frac{-3}{8} & \frac{4}{9} & \frac{-5}{10} & \frac{6}{11} & \frac{-7}{12} & \frac{8}{13} & \frac{-9}{14} & \cdots \\
\frac{1}{6} & \frac{-1}{7} & \frac{2}{8} & \frac{-3}{9} & \frac{4}{10} & \frac{-5}{11} & \frac{6}{12} & \frac{-7}{13} & \frac{8}{14} & \frac{-9}{15} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{array}
\]

Next, draw an infinite path in this array, beginning at $\frac{0}{1}$ and snaking back and forth as indicated below. Every rational number is on this path.
Beginning at \(\{\frac{0}{1}\}\) and following the path, we get an infinite list of all rational numbers:

\[
0, 1, \frac{1}{2}, -\frac{1}{2}, -1, 2, \frac{2}{3}, \frac{2}{5}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, \frac{2}{7}, -\frac{2}{7}, -\frac{2}{5}, -\frac{2}{3}, -2, 3, \frac{3}{2}, \frac{3}{4}, \frac{3}{5}, \cdots
\]

By Theorem 14.3, it follows that \(\mathbb{Q}\) is countably infinite, that is, \(|\mathbb{Q}| = |\mathbb{N}|\).

It is also true that the Cartesian product of two countably infinite sets is itself countably infinite, as our next theorem states.

**Theorem 14.5**

If A and B are both countably infinite, then so is \(A \times B\).

**Proof.** Suppose A and B are both countably infinite. By Theorem 14.3, we know we can write A and B in list form as

\[
A = \{a_1, a_2, a_3, a_4, \ldots\},
\]

\[
B = \{b_1, b_2, b_3, b_4, \ldots\}.
\]

Figure 14.2 shows how to form an infinite path winding through all of \((A \times B)\).

Therefore \((A \times B)\) can be written in list form, so it is countably infinite.
Figure 14.2. A product of two countably infinite sets is countably infinite

As an example of a consequence of this theorem, notice that since \(\mathbb{Q}\) is countably infinite, the set \(\mathbb{Q} \times \mathbb{Q}\) is also countably infinite. Recall that the word “corollary” means a result that follows easily from some other result. We have the following corollary of Theorem 14.5.

Corollary 14.1

Given \(n\) countably infinite sets \(A_1, A_2, \cdots, A_n\), with \(n \ge 2\), the Cartesian product \(A_1 \times A_2 \times \cdots \times A_n\) is also countably infinite.

Proof. The proof is by induction on \(n\). For the basis step, notice that when \(n = 2\) the statement asserts that for countably infinite sets \(A_1\) and \(A_2\), the product \(A_1 \times A_2\) is countably infinite, and this is true by Theorem 14.5.

Assume that for some \(k \ge 2\), any product \(A_1 \times A_2 \times \cdots \times A_k\) of countably infinite sets is countably infinite. Consider a product \(A_1 \times A_2 \times \cdots \times A_k \times A_{k+1}\) of \(k+1\) countably infinite sets. It is easy to confirm that the function

\[
f : A_1 \times A_2 \times \cdots \times A_k \times A_{k+1} \rightarrow (A_1 \times A_2 \times \cdots \times A_k) \times A_{k+1}
\]

\[
f((x_1, x_2, \cdots, x_k, x_{k+1})) = ((x_1, x_2, \cdots, x_k), x_{k+1})
\]

is bijective, so \(|A_1 \times A_2 \times \cdots \times A_k \times A_{k+1}| = |(A_1 \times A_2 \times \cdots \times A_k) \times A_{k+1}| = |A_1 \times A_2 \times \cdots \times A_{k+1}|\).
By the induction hypothesis, \((A_1 \times A_2 \times \cdots \times A_k) \times A_{k+1}\) is a product of two countably infinite sets, so it is countably infinite by Theorem 14.5. As noted above, \((A_1 \times A_2 \times \cdots \times A_k) \times A_{k+1}\) has the same cardinality as the set \((A_1 \times A_2 \times \cdots \times A_k) \times A_{k+1}\), so it too is countably infinite.

**Theorem 14.6**

If A and B are both countably infinite, then their union \((A \cup B)\) is countably infinite.

**Proof.** Suppose \(A\) and \(B\) are both countably infinite. By Theorem 14.3, we know we can write \(A\) and \(B\) in list form as

\[
A = \{a_1, a_2, a_3, a_4, \ldots\},
\]

\[
B = \{b_1, b_2, b_3, b_4, \ldots\},
\]

We can “shuffle” \(A\) and \(B\) into one infinite list for \((A \cup B)\) as follows.

\[
(A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, \ldots\},
\]

(We agree not to list an element twice if it belongs to both \(A\) and \(B\).) Thus \((A \cup B)\) is countably infinite by Theorem 14.3.

**Exercise**

Exercise \(\PageIndex{1}\)

Prove that the set \((A = \{In(n) : n \in \mathbb{N}\} \subseteq \mathbb{R}\) is countably infinite.

Exercise \(\PageIndex{2}\)

Prove that the set \((A = \{(m, n) : m \le n\} \subseteq \mathbb{N} \times \mathbb{N}\) is countably infinite.

Exercise \(\PageIndex{3}\)

Prove that the set \((A = \{(5n, -3n) : n \in \mathbb{Z}\})\) is countably infinite.

Exercise \(\PageIndex{4}\)

Prove that the set of all irrational numbers is uncountable. (Suggestion: Consider proof by contradiction using Theorems 14.4 and 14.6.)

Exercise \(\PageIndex{5}\)

Prove or disprove: There exists a countably infinite subset of the set of irrational numbers.

Exercise \(\PageIndex{6}\)
Prove or disprove: There exists a bijective function \( f : \mathbb{Q} \rightarrow \mathbb{R} \).

Exercise \( \PageIndex{7} \)

Prove or disprove: The set \( \mathbb{Q}^{100} \) is countably infinite.

Exercise \( \PageIndex{8} \)

Prove or disprove: The set \( \mathbb{Z} \times \mathbb{Q} \) is countably infinite.

Exercise \( \PageIndex{9} \)

Prove or disprove: The set \( \{0,1\} \times \mathbb{N} \) is countably infinite.

Exercise \( \PageIndex{10} \)

Prove or disprove: The set \( A = \{ \frac{\sqrt{2}}{n} : n \in \mathbb{N} \} \) is countably infinite.

Exercise \( \PageIndex{11} \)

Describe a partition of \( \mathbb{N} \) that divides \( \mathbb{N} \) into eight countably infinite subsets.

Exercise \( \PageIndex{12} \)

Describe a partition of \( \mathbb{N} \) that divides \( \mathbb{N} \) into \( \aleph_0 \) countably infinite subsets.

Exercise \( \PageIndex{13} \)

Prove or disprove: If \( A = \{ X \subseteq \mathbb{N} : X \text{ is finite} \} \), then \( |A| = \aleph_0 \).

Exercise \( \PageIndex{14} \)

Suppose \( A= \{(m,n) \in \mathbb{N} \times \mathbb{R} : n = \pi m \} \). Is it true that \( |\mathbb{N}| = |A| \)?

Exercise \( \PageIndex{15} \)

Theorem 14.5 implies that \( \mathbb{N} \times \mathbb{N} \) is countably infinite. Construct an alternate proof of this fact by showing that the function \( \varphi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) defined as \( \varphi (m,n) = 2^{n-1}(2m-1) \) is bijective.