7.1: Invariant Subspaces

To begin our study, we will look at subspaces \(U\) of \(V\) that have special properties under an operator \(T\) in \(\mathcal{L}(V,V)\).

Definition \(\PageIndex{1}\): invariant subspace

Let \(V\) be a finite-dimensional vector space over \(\mathbb{F}\) with \(\dim(V) \geq 1\), and let \(T \in \mathcal{L}(V,V)\) be an operator in \(V\). Then a subspace \(U \subset V\) is called an invariant subspace under \(T\) if

\[
Tu \in U \quad \text{for all } u \in U.
\]

That is, \(U\) is invariant under \(T\) if the image of every vector in \(U\) under \(T\) remains within \(U\). We denote this as \(TU = \{ Tu \mid u \in U \} \subset U\).

Example \(\PageIndex{1}\)

The subspaces \(\ker(T)\) and \(\text{range}(T)\) are invariant subspaces under \(T\). To see this, let \(u \in \ker(T)\). This means that \(Tu = 0\). But, since \(0 \in \ker(T)\), this implies that \(Tu = 0 \in \ker(T)\). Similarly, let \(u \in \text{range}(T)\). Since \(Tv \in \text{range}(T)\) for all \(v \in V\), we certainly also have that \(Tu \in \text{range}(T)\).

Example \(\PageIndex{2}\)

Take the linear operator \(T: \mathbb{R}^3 \to \mathbb{R}^3\) corresponding to the matrix
with respect to the basis \(((e_1,e_2,e_3))\). Then \((\text{Span}(e_1,e_2))\) and \((\text{Span}(e_3))\) are both invariant subspaces under \((T)\).

An important special case of Definition 7.1.1 involves one-dimensional invariant subspaces under an operator \((T)\) in \((\text{L}(V,V))\). If \((\dim(U) = 1)\), then there exists a nonzero vector \((u)\) in \((V)\) such that

\[
U = \{ au \mid a \in \mathbb{F} \}.
\]

In this case, we must have

\[
T u = \lambda u \quad \text{for some } \lambda \in \mathbb{F}.
\]

This motivates the definitions of eigenvectors and eigenvalues of a linear operator, as given in the next section.

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