5.7: Integrals Resulting in Inverse Trigonometric Functions

Learning Objectives

• Integrate functions resulting in inverse trigonometric functions

In this section we focus on integrals that result in inverse trigonometric functions. We have worked with these functions before. Recall, that trigonometric functions are not one-to-one unless the domains are restricted. When working with inverses of trigonometric functions, we always need to be careful to take these restrictions into account. Also, we previously developed formulas for derivatives of inverse trigonometric functions. The formulas developed there give rise directly to integration formulas involving inverse trigonometric functions.

Integrals that Result in Inverse Trigonometric Functions

Let us begin this last section of the chapter with the three formulas. Along with these formulas, we use substitution to evaluate the integrals. We prove the formula for the inverse sine integral.

Rule: Integration Formulas Resulting in Inverse Trigonometric Functions

The following integration formulas yield inverse trigonometric functions:

\[
\begin{align}
\int \dfrac{du}{\sqrt{a^2-u^2}} &= \sin^{-1}\left(\dfrac{u}{a}\right) + C \\
\int \dfrac{du}{a^2+u^2} &= \dfrac{1}{a} \tan^{-1}\left(\dfrac{u}{a}\right) + C \\
\int \dfrac{du}{u\sqrt{u^2-a^2}} &= \dfrac{1}{a} \sec^{-1}\left(\dfrac{|u|}{a}\right) + C
\end{align}
\]

Proof of the first formula
Let \( y = \sin^{-1}\left(\frac{x}{a}\right) \). Then \( a \sin y = x \). Now using implicit differentiation, we obtain

\[
\frac{d}{dx}(a \sin y) = \frac{d}{dx}(x)
\]

\[
a \cos y \frac{dy}{dx} = 1
\]

\[
\frac{dy}{dx} = \frac{1}{a \cos y}.
\]

For \( -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \cos y \geq 0 \) Thus, applying the Pythagorean identity \( \sin^2 y + \cos^2 y = 1 \), we have \( \cos y = \sqrt{1 - \sin^2 y} \). This gives

\[
\begin{align}
\frac{1}{a \cos y} &= \frac{1}{a \sqrt{1 - \sin^2 y}} \\
&= \frac{1}{\sqrt{a^2 - a^2 \sin^2 y}} \\
&= \frac{1}{\sqrt{a^2 - x^2}}.
\end{align}
\]

Then for \( -a \leq x \leq a \) we have

\[
\int \frac{1}{\sqrt{a^2 - u^2}} 
\]

\[
\begin{align}
\text{\text{\text{\begin{align}}}}
\frac{1}{a \sqrt{1 - \sin^2 y}} \\
= \frac{1}{\sqrt{a^2 - a^2 \sin^2 y}} \\
= \frac{1}{\sqrt{a^2 - x^2}}.
\end{align}
\]

Example \( \PageIndex{1} \): Evaluating a Definite Integral Using Inverse Trigonometric Functions

Evaluate the definite integral

\[
\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}.
\]

Solution

We can go directly to the formula for the antiderivative in the rule on integration formulas resulting in inverse trigonometric functions, and then evaluate the definite integral. We have

\[
\begin{align}
\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} x \bigg|_0^{1/2} \\
&= \sin^{-1} \frac{1}{2} - \sin^{-1} 0 \\
&= \frac{\pi}{6}.
\end{align}
\]

Note that since the integrand is simply the derivative of \( \sin^{-1} x \), we are really just using this fact to find the antiderivative here.

Exercise \( \PageIndex{1} \)

Find the indefinite integral using an inverse trigonometric function and substitution for \( \displaystyle \int \frac{dx}{\sqrt{9-x^2}} \).

Hint

Use the formula in the rule on integration formulas resulting in inverse trigonometric functions.
In many integrals that result in inverse trigonometric functions in the antiderivative, we may need to use substitution to see how to use the integration formulas provided above.

Example 2: Finding an Antiderivative Involving an Inverse Trigonometric Function using substitution

Evaluate the integral
\[ \int \frac{dx}{\sqrt{4-9x^2}}. \]

Solution
Substitute \( u = 3x \). Then \( du = 3 \, dx \) and we have
\[ \int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3} \int \frac{du}{\sqrt{4-u^2}}. \]

Applying the formula with \( a = 2 \), we obtain
\[ \int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3} \sin^{-1} \left( \frac{u}{2} \right) + C = \frac{1}{3} \sin^{-1} \left( \frac{3x}{2} \right) + C. \]

Exercise 2
Find the antiderivative of \( \int \frac{dx}{\sqrt{1-16x^2}}. \)

Hint
Substitute \( u = 4x \).

Answer
\( \int \frac{dx}{\sqrt{1-16x^2}} = \frac{1}{4} \sin^{-1}(4x) + C \)

Example 3: Evaluating a Definite Integral

Evaluate the definite integral
\[ \int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}}. \]

Solution
The format of the problem matches the inverse sine formula. Thus,
$\int_{0}^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u \bigg|_{0}^{\sqrt{3}/2} = \sin^{-1} \left( \frac{\sqrt{3}}{2} \right) - \sin^{-1}(0) = \frac{\pi}{3}$.

Integrals Resulting in Other Inverse Trigonometric Functions

There are six inverse trigonometric functions. However, only three integration formulas are noted in the rule on integration formulas resulting in inverse trigonometric functions because the remaining three are negative versions of the ones we use. The only difference is whether the integrand is positive or negative. Rather than memorizing three more formulas, if the integrand is negative, simply factor out $-1$ and evaluate the integral using one of the formulas already provided. To close this section, we examine one more formula: the integral resulting in the inverse tangent function.

Example $\PageIndex{4}$: Finding an Antiderivative Involving the Inverse Tangent Function

Find the antiderivative of $\int \frac{1}{9+x^2} \, dx$.

**Solution**

Apply the formula with $a=3$. Then,

$$\int \frac{dx}{9+x^2} = \frac{1}{3} \tan^{-1} \left( \frac{x}{3} \right) + C.$$

**Exercise $\PageIndex{3}$**

Find the antiderivative of $\int \frac{dx}{16+x^2}$.

**Hint**

Follow the steps in Example $\PageIndex{4}$.

**Answer**

$$\int \frac{dx}{16+x^2} = \frac{1}{4} \tan^{-1} \left( \frac{x}{4} \right) + C$$

Example $\PageIndex{5}$: Applying the Integration Formulas WITH SUBSTITUTION

Find an antiderivative of $\int \frac{1}{1+4x^2} \, dx$.

**Solution**

Comparing this problem with the formulas stated in the rule on integration formulas resulting in inverse trigonometric functions, the integrand looks similar to the formula for $\tan^{-1}(-1) + C$. So we use substitution, letting $u=2x$, then $du=2 \, dx$.

Then, we have

$$\int \frac{du}{1+u^2} = \tan^{-1}(-1) + C.$$
Exercise \( \PageIndex{4} \)

Use substitution to find the antiderivative of \( \displaystyle \int \frac{dx}{25+4x^2} \).

**Hint**

Use the solving strategy from Example \( \PageIndex{5} \) and the rule on integration formulas resulting in inverse trigonometric functions.

**Answer**

\( \displaystyle \int \frac{dx}{25+4x^2} = \frac{1}{10} \tan^{-1}\left(\frac{2x}{5}\right)+C \)

Example \( \PageIndex{6} \): Evaluating a Definite Integral

Evaluate the definite integral \( \displaystyle \int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2} \).

**Solution**

Use the formula for the inverse tangent. We have

\[
\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}.
\]

Exercise \( \PageIndex{5} \)

Evaluate the definite integral \( \displaystyle \int_{0}^{2} \frac{dx}{4+x^2} \).

**Hint**

Follow the procedures from Example \( \PageIndex{6} \) to solve the problem.

**Answer**

\( \displaystyle \int_{0}^{2} \frac{dx}{4+x^2} = \frac{\pi}{8} \)

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**Key Concepts**

- Formulas for derivatives of inverse trigonometric functions developed in Derivatives of Exponential and Logarithmic Functions lead directly to integration formulas involving inverse trigonometric functions.
- Use the formulas listed in the rule on integration formulas resulting in inverse trigonometric functions to match up the correct format and make alterations as necessary to solve the problem.
- Substitution is often required to put the integrand in the correct form.
**Key Equations**

- **Integrals That Produce Inverse Trigonometric Functions**
  
  \[
  \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C
  \]

  \[
  \int \frac{du}{a^2+u^2} = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C
  \]

  \[
  \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} \sec^{-1}\left(\frac{|u|}{a}\right) + C
  \]

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