5.9: The General Solution of a Linear System

Learning Objectives

1. Use linear transformations to determine the particular solution and general solution to a system of equations.
2. Find the kernel of a linear transformation.

Recall the definition of a linear transformation discussed above. \( T \) is a **linear transformation** if whenever \( \vec{x}, \vec{y} \) are vectors and \( k, p \) are scalars, \( T(k\vec{x} + p\vec{y}) = kT(\vec{x}) + pT(\vec{y}) \). Thus linear transformations distribute across addition and pass scalars to the outside.

It turns out that we can use linear transformations to solve linear systems of equations. Indeed given a system of linear equations of the form \( A\vec{x} = \vec{b} \), one may rephrase this as \( T(\vec{x}) = \vec{b} \) where \( T \) is the linear transformation \( T_A \) induced by the coefficient matrix \( A \). With this in mind consider the following definition.

**Definition**

Particular Solution of a System of Equations

Suppose a linear system of equations can be written in the form \( T(\vec{x}) = \vec{b} \). If \( T(\vec{x}) = \vec{b} \) then \( \vec{x} \) is called a **particular solution** of the linear system.

Recall that a system is called homogeneous if every equation in the system is equal to \( 0 \). Suppose we represent a homogeneous system of equations by \( T(\vec{x}) = \vec{0} \). It turns out that the \( \vec{x} \) for which \( T(\vec{x}) = \vec{0} \) are part of a special set called the **null space** of \( T \). We may also refer to the null space as the **kernel** of \( T \), and we write \( ker(T) \).
Definition \(\PageIndex{2}\): Null Space or Kernel of a Linear Transformation

Let \(T\) be a linear transformation. Define \(\ker (T) = \{ \vec{x} : T(\vec{x}) = \vec{0} \}\) The kernel, \(\ker (T)\) consists of the set of all vectors \(\vec{x}\) for which \(T(\vec{x}) = \vec{0}\). This is also called the null space of \(T\).

We may also refer to the kernel of \(T\) as the solution space of the equation \(T(\vec{x}) = \vec{0}\).

Consider the following example.

Example \(\PageIndex{1}\): The Kernel of the Derivative

Let \(\frac{d}{dx}\) denote the linear transformation defined on \(f\), the functions which are defined on \(\mathbb{R}\) and have a continuous derivative. Find \(\ker (\frac{d}{dx})\).

Solution

The example asks for functions \(f\) which the property that \(\frac{df}{dx} = 0\). As you may know from calculus, these functions are the constant functions. Thus \(\ker (\frac{d}{dx})\) is the set of constant functions.

Definition \[\text{def:nullspaceoflineartransformation}\] states that \(\ker (T)\) is the set of solutions to the equation, \(T(\vec{x}) = \vec{0}\) Since we can write \(T(\vec{x}) = A\vec{x}\), you have been solving such equations for quite some time.

We have spent a lot of time finding solutions to systems of equations in general, as well as homogeneous systems. Suppose we look at a system given by \(A\vec{x} = \vec{b}\), and consider the related homogeneous system. By this, we mean that we replace \(\vec{b}\) by \(\vec{0}\) and look at \(A\vec{x} = \vec{0}\). It turns out that there is a very important relationship between the solutions of the original system and the solutions of the associated homogeneous system. In the following theorem, we use linear transformations to denote a system of equations. Remember that \(T(\vec{x})\) is the set of constant functions.

Theorem \(\PageIndex{1}\): Particular Solution and General Solution

Suppose \(\{\vec{x}_p\}_p\) is a solution to the linear system given by, \(T(\vec{x}) = \vec{b}\) Then if \(\{\vec{y}\}_p\) is any other solution to \(T(\vec{x}) = \vec{b}\), there exists \(\{\vec{x}_0\}_p\) in \(\ker (T)\) such that \(\vec{y} = \vec{x}_p + \vec{x}_0\) Hence, every solution to the linear system can be written as a sum of a particular solution, \(\{\vec{x}_p\}_p\), and a solution \(\{\vec{x}_0\}_p\) to the associated homogeneous system given by \(T(\vec{x}) = \vec{0}\).

Proof

Consider \(\vec{y} - \vec{x}_p = \vec{y} + \vec{x}_0\) Then \(T(\vec{y} + \vec{x}_0) = T(\vec{y}) + T(\vec{x}_0) = \vec{b} + \vec{0} = \vec{b}\) Hence, \(\vec{y} = \vec{x}_p + \vec{x}_0\) for all \(\vec{y}\) and \(\vec{x}_p\) are both solutions to the system, it follows that \(T(\vec{y} + \vec{x}_0) = \vec{b}\) and \(T(\vec{x}_p + \vec{x}_0) = \vec{b}\).

Hence, \(\vec{y} = \vec{x}_p + \vec{x}_0\) Let \(\vec{x}_0 = \vec{y} - \vec{x}_p\)
Then, \( T(\vec{x}_0) = \vec{0} \) so \( \vec{x}_0 \) is a solution to the associated homogeneous system and so is in \( \ker(T) \).

Sometimes people remember the above theorem in the following form. The solutions to the system \( T(\vec{x}) = \vec{b} \) are given by \( \vec{x}_p + \ker(T) \) where \( \vec{x}_p \) is a particular solution to \( T(\vec{x}) = \vec{b} \).

For now, we have been speaking about the kernel or null space of a linear transformation \( T \). However, we know that every linear transformation \( T \) is determined by some matrix \( A \). Therefore, we can also speak about the null space of a matrix. Consider the following example.

Example \( \PageIndex{2} \): The Null Space of a Matrix

Let \( A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix} \) Find \( \text{null}(A) \). Equivalently, find the solutions to the system of equations \( A\vec{x} = \vec{0} \).

**Solution**

We are asked to find \( \{ \vec{x} : A\vec{x} = \vec{0} \} \). In other words we want to solve the system, \( A\vec{x} = \vec{0} \). Let \( \vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \) Then this amounts to solving \( A\vec{x} = \vec{0} \) for \( \vec{x} \).

This is the linear system \( A\vec{x} = \vec{0} \). To solve, set up the augmented matrix and row reduce to find the reduced row-echelon form.

\[
\begin{bmatrix}
1 & 2 & 3 & 0 & 0 \\
2 & 1 & 1 & 2 & 0 \\
4 & 5 & 7 & 2 & 0
\end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix}
1 & 0 & -\frac{1}{3} & \frac{4}{3} & 0 \\
0 & 1 & \frac{5}{3} & -\frac{2}{3} & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

This yields \( x = \frac{1}{3}z - \frac{4}{3}w \) and \( y = \frac{2}{3}w - \frac{5}{3}z \). Since \( \text{null}(A) \) consists of the solutions to this system, it consists vectors of the form,

\[
\begin{bmatrix}
\frac{1}{3} \\
-\frac{5}{3} \\
1 \\
0
\end{bmatrix}
\]

This yields \( x = \frac{1}{3}z - \frac{4}{3}w \) and \( y = \frac{2}{3}w - \frac{5}{3}z \). Since \( \text{null}(A) \) consists of the solutions to this system, it consists vectors of the form,

\[
\begin{bmatrix}
\frac{1}{3} \\
-\frac{5}{3} \\
1 \\
0
\end{bmatrix}
\]

Consider the following example.

Example \( \PageIndex{3} \): A General Solution

The **general solution** of a linear system of equations is the set of all possible solutions. Find the general solution to the linear system,
\[
\left( \begin{array}{rrrr}
1 & 2 & 3 & 0 \\
2 & 1 & 1 & 2 \\
4 & 5 & 7 & 2 \\
\end{array} \right)
\left( \begin{array}{r}
x \\
y \\
z \\
w \\
\end{array} \right) =
\left( \begin{array}{r}
9 \\
7 \\
25 \\
\end{array} \right)
\]

given that \(\left( \begin{array}{r}
x \\
y \\
z \\
w \\
\end{array} \right) = \left( \begin{array}{r}
1 \\
1 \\
2 \\
1 \\
\end{array} \right)\) is one solution.

**Solution**

Note the matrix of this system is the same as the matrix in Example [exa:matrixnullspace]. Therefore, from Theorem [thm:particularandgeneralsolution], you will obtain all solutions to the above linear system by adding a particular solution \((\vec{x}_p)\) to the solutions of the associated homogeneous system, \((\vec{x})\). One particular solution is given above by

\[
\vec{x}_p = \left( \begin{array}{r}
x \\
y \\
z \\
w \\
\end{array} \right) = \left( \begin{array}{r}
1 \\
1 \\
2 \\
1 \\
\end{array} \right)
\]

Using this particular solution along with the solutions found in Example [exa:matrixnullspace], we obtain the following solutions,

\[
\left( \begin{array}{r}
\frac{1}{3} \\
-\frac{5}{3} \\
1 \\
0 \\
\end{array} \right) + w \left( \begin{array}{r}
-\frac{4}{3} \\
\frac{2}{3} \\
0 \\
1 \\
\end{array} \right) + \left( \begin{array}{r}
1 \\
1 \\
2 \\
1 \\
\end{array} \right)
\]

Hence, any solution to the above linear system is of this form.