4.4: Sums and direct sum

Throughout this section, \( V \) is a vector space over \( \mathbb{F} \), and \( U_1, U_2 \subset V \) denote subspaces.

Definition 4.4.1: (subspace) sum

Let \( U_1, U_2 \subset V \) be subspaces of \( V \). Define the **subspace sum** of \( U_1 \)

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Check as an exercise that \((U_1 + U_2)\) is a subspace of \((V)\). In fact, \((U_1 + U_2)\) is the smallest subspace of \((V)\) that contains both \((U_1)\) and \((U_2)\).

**Example 4.4.2.** Let

\[
\begin{align*}
U_1 &= \{(x, 0, 0) \in \mathbb{F}^3 | x \in \mathbb{F}\}, \\
U_2 &= \{(0, y, 0) \in \mathbb{F}^3 | y \in \mathbb{F}\}.
\end{align*}
\]

Then

\[
U_1 + U_2 = \{(x, y, 0) \in \mathbb{F}^3 | x, y \in \mathbb{F}\}. \tag{4.4.2}
\]

If, alternatively, \(U_2 = \{(y, y, 0) \in \mathbb{F}^3 | y \in \mathbb{F}\}\), then Equation (4.4.2) still holds.

If \((U = U_1 + U_2)\), then, for any \((u \in U)\), there exist \((u_1 \in U_1)\) and \((u_2 \in U_2)\) such that \((u = u_1 + u_2)\).

If it so happens that \((u)\) can be uniquely written as \((u_1 + u_2)\), then \((U)\) is called the **direct sum** of \((U_1)\) and \((U_2)\).

**Definition 4.4.3**: Direct Sum

Suppose every \((u \in U)\) can be uniquely written as \((u = u_1 + u_2)\) for \((u_1 \in U_1)\) and \((u_2 \in U_2)\). Then we use

\[
[U = U_1 \oplus U_2]
\]

to denote the **direct sum** of \((U_1)\) and \((U_2)\).

**Example 4.4.4.** Let

\[
\begin{align*}
U_1 &= \{(x, y, 0) \in \mathbb{F}^3 | x, y \in \mathbb{F}\}, \\
U_2 &= \{(0, 0, z) \in \mathbb{F}^3 | z \in \mathbb{F}\}.
\end{align*}
\]

Then \(\mathbb{R}^3 = U_1 \oplus U_2\). However, if instead

\[
U_2 = \{(0, w, z) | w, z \in \mathbb{F}\},
\]

then \(\mathbb{R}^3 = U_1 + U_2\) but is not the direct sum of \((U_1)\) and \((U_2)\).

**Example 4.4.5.** Let

\[
\begin{align*}
U_1 &= \{p \in \mathbb{F}[z] | p(z) = a_0 + a_2 z^2 + \cdots + a_{2m} z^{2m} \}, \\
U_2 &= \{p \in \mathbb{F}[z] | p(z) = a_1z + a_3z^3 + \cdots + a_{2m+1} z^{2m+1} \}.
\end{align*}
\]

Then \(\mathbb{F}[z] = U_1 \oplus U_2\).

**Proposition 4.4.6.** Let \((U_1, U_2 \subseteq V)\) be subspaces. Then \((V = U_1 \oplus U_2)\) if and only if the following two conditions hold:

1. \((V = U_1 + U_2)\)
2. If \((0 = u_1 + u_2)\) with \((u_1 \in U_1)\) and \((u_2 \in U_2)\), then \((u_1 = u_2 = 0)\).

**Proof.**
Suppose \( V = U_1 \oplus U_2 \). Then Condition 1 holds by definition. Certainly \( 0 = 0 + 0 \), and, since by uniqueness this is the only way to write \( 0 \in V \), we have \( u_1 = u_2 = 0 \).

Suppose Conditions 1 and 2 hold. By Condition 1, we have that, for all \( (v \in V) \), there exist \( (u_1 \in U_1) \) and \( (u_2 \in U_2) \) such that \( (v = u_1 + u_2) \). Suppose \( (v = w_1 + w_2) \) with \( (w_1 \in U_1) \) and \( (w_2 \in U_2) \). Subtracting the two equations, we obtain

\[
[0 = (u_1 - w_1) + (u_2 - w_2)],
\]

where \( (u_1 - w_1 \in U_1) \) and \( (u_2 - w_2 \in U_2) \). By Condition 2, this implies \( (u_1 - w_1 = 0) \) and \( (u_2 - w_2 = 0) \), or equivalently \( (u_1 = w_1) \) and \( (u_2 = w_2) \), as desired.

**Proposition 4.4.7.** Let \( (U_1, U_2 \subset V) \) be subspaces. Then \( (V = U_1 \oplus U_2) \) if and only if the following two conditions hold:

1. \( (V = U_1 + U_2) \)
2. \( (U_1 \cap U_2 = \{0\}) \)

**Proof.**

Suppose \( (V = U_1 \oplus U_2) \). Then Condition 1 holds by definition. If \( (u \in U_1 \cap U_2) \), then \( 0 = u + (−u) \) with \( (u \in U_1) \) and \( (−u \in U_2) \) (why?). By Proposition 4.4.6, we have \( (u = 0) \) and \( (−u = 0) \) so that \( (U_1 \cap U_2 = \{0\}) \).

Suppose Conditions 1 and 2 hold. To prove that \( (V = U_1 \oplus U_2) \) holds, suppose that

\[
[0 = u_1 + u_2, \forall \sim \text{where~} u_1 \in U_1 \forall \sim \text{and~} u_2 \in U_2. \tag{4.3}]
\]

By Proposition 4.4.6, it suffices to show that \( (u_1 = u_2 = 0) \). Equation (4.3) implies that \( (u_1 = −u_2 \in U_2) \). Hence \( (u_1 \in U_1 \cap U_2) \), which in turn implies that \( (u_1 = 0) \). It then follows that \( (u_2 = 0) \) as well.

Everything in this section can be generalized to \( m \) subspaces \( (U_1, U_2, \ldots U_m) \) with the notable exception of Proposition 4.4.7. To see, this consider the following example.

**Example 4.4.8.** Let

\[
U_1 = \{(x, y, 0) \in \mathbb{F}^3 | x, y \in \mathbb{F}\}, \ln U_2 = \{(0, 0, z) \in \mathbb{F}^3 | z \in \mathbb{F}\}, \ln U_3 = \{(0, y, y) \in \mathbb{F}^3 | y \in \mathbb{F}\}.
\]

Then certainly \( (\mathbb{F}^3 = U_1 + U_2 + U_3) \), but \( (\mathbb{F}^3 \neq U_1 \oplus U_2 \oplus U_3) \) since, for example,

\[
[(0, 0, 0) = (0, 1, 0) + (0, 0, 1) + (0, −1, −1)].
\]

But \( (U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{0\}) \) so that the analog of Proposition 4.4.7 does not hold.
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