4.4: Sums and direct sum

Throughout this section, \( (V) \) is a vector space over \( (\mathbb{F}) \), and \( (U_1 \cup U_2 \subset V) \) denote subspaces.

Definition 4.4.1: (subspace) sum

Let \( (U_1 \cup U_2 \subset V) \) be subspaces of \( (V) \). Define the (subspace) sum of \( (U_1 \cup U_2) \) to be the set

![Diagram](image)

**Figure 4.4.1:** The union \( (U \cup U') \) of two subspaces is not necessarily a subspace.

and \( (U_2 \cup U') \) to be the set
\[ \{ u_1 + u_2 \mid u_1 \in U_1, u_2 \in U_2 \}. \tag{4.4.1} \]

Check as an exercise that \((U_1 + U_2)\) is a subspace of \((V)\). In fact, \((U_1 + U_2)\) is the smallest subspace of \((V)\) that contains both \((U_1)\) and \((U_2)\).

**Example 4.4.2.** Let
\[ U_1 = \{(x, 0, 0) \in \mathbb{F}^3 \mid x \in \mathbb{F}\}, \quad U_2 = \{(0, y, 0) \in \mathbb{F}^3 \mid y \in \mathbb{F}\}. \]
Then
\[ U_1 + U_2 = \{(x, y, 0) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}. \tag{4.4.2} \]
If, alternatively, \((U_2 = \{(y, y, 0) \in \mathbb{F}^3 \mid y \in \mathbb{F}\})\), then Equation (4.4.2) still holds.

If \((U = U_1 + U_2)\), then, for any \((u \in U)\), there exist \((u_1 \in U_1)\) and \((u_2 \in U_2)\) such that \((u = u_1 + u_2)\).

If it so happens that \((u)\) can be uniquely written as \((u_1 + u_2)\), then \((U)\) is called the **direct sum** of \((U_1)\) and \((U_2)\).

**Definition 4.4.3: Direct Sum**
Suppose every \((u \in U)\) can be uniquely written as \((u = u_1 + u_2)\) for \((u_1 \in U_1)\) and \((u_2 \in U_2)\). Then we use
\[ U = U_1 \oplus U_2 \]
to denote the **direct sum** of \((U_1)\) and \((U_2)\).

**Example 4.4.4.** Let
\[ U_1 = \{(x, y, 0) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}, \quad U_2 = \{(0, 0, z) \in \mathbb{F}^3 \mid z \in \mathbb{F}\}. \]
Then \((\mathbb{F}^3 = U_1 \oplus U_2)\). However, if instead
\[ U_2 = \{(0, w, z) \mid w, z \in \mathbb{F}\}, \]
then \((\mathbb{F}^3 = U_1 + U_2)\) but is not the direct sum of \((U_1)\) and \((U_2)\).

**Example 4.4.5.** Let
\[ U_1 = \{p \in \mathbb{F}\lbrack z \rbrack \mid p(z) = a_0 + a_2 z^2 + \cdots + a_{2m} z^{2m} \}, \quad U_2 = \{p \in \mathbb{F}\lbrack z \rbrack \mid p(z) = a_1 z + a_3 z^3 + \cdots + a_{2m+1} z^{2m+1} \}. \]
Then \((\mathbb{F}\lbrack z \rbrack = U_1 \oplus U_2)\).

**Proposition 4.4.6.** Let \((U_1, U_2 \subset V)\) be subspaces. Then \((V = U_1 \oplus U_2)\) if and only if the following two conditions hold:

1. \((V = U_1 + U_2)\)
2. \(\{0 = u_1 + u_2\} \text{ with } (u_1 \in U_1) \text{ and } (u_2 \in U_2)\), then \((u_1 = u_2 = 0)\).
Proof.

("\Rightarrow") Suppose \(V = U_1 \oplus U_2\). Then Condition 1 holds by definition. Certainly \(0 = 0 + 0\), and, since by uniqueness this is the only way to write \(0 \in V\), we have \(u_1 = u_2 = 0\).

("\Leftarrow") Suppose Conditions 1 and 2 hold. By Condition 1, we have that, for all \(v \in V\), there exist \(u_1 \in U_1\) and \(u_2 \in U_2\) such that \(v = u_1 + u_2\). Suppose \(v = w_1 + w_2\) with \(w_1 \in U_1\) and \(w_2 \in U_2\). Subtracting the two equations, we obtain

\[0 = (u_1 - w_1) + (u_2 - w_2),\]

where \((u_1 - w_1) \in U_1\) and \((u_2 - w_2) \in U_2\). By Condition 2, this implies \((u_1 - w_1 = 0)\) and \((u_2 - w_2 = 0)\), or equivalently \((u_1 = w_1)\) and \((u_2 = w_2)\), as desired.

Proposition 4.4.7. Let \(U_1, U_2 \subset V\) be subspaces. Then \(V = U_1 \oplus U_2\) if and only if the following two conditions hold:

1. \(V = U_1 + U_2\);
2. \(U_1 \cap U_2 = \{0\}\).

Proof.

("\Rightarrow") Suppose \(V = U_1 \oplus U_2\). Then Condition 1 holds by definition. If \((u \in U_1 \cap U_2)\), then \(0 = u + (−u)\) with \(u \in U_1\) and \(−u \in U_2\) (why?). By Proposition 4.4.6, we have \((u = 0)\) and \((−u = 0)\) so that \((U_1 \cap U_2 = \{0\})\).

("\Leftarrow") Suppose Conditions 1 and 2 hold. To prove that \(V = U_1 \oplus U_2\) holds, suppose that \(0 = u_1 + u_2\), where \(u_1 \in U_1\) and \(u_2 \in U_2\). Equation (4.3) implies that \(u_1 = −u_2 \in U_2\). Hence \((u_1 \in U_1 \cap U_2)\), which in turn implies that \((u_1 = 0)\). It then follows that \((u_2 = 0)\) as well.

Everything in this section can be generalized to m subspaces \((U_1, U_2, \ldots U_m)\) with the notable exception of Proposition 4.4.7. To see, this consider the following example.

Example 4.4.8. Let

\[
\begin{align*}
U_1 &= \{(x, y, 0) \in \mathbb{F}^3 | x, y \in \mathbb{F}\}, \\
U_2 &= \{(0, 0, z) \in \mathbb{F}^3 | z \in \mathbb{F}\}, \\
U_3 &= \{(0, y, y) \in \mathbb{F}^3 | y \in \mathbb{F}\}.
\end{align*}
\]

Then certainly \(\mathbb{F}^3 = U_1 + U_2 + U_3\), but \(\mathbb{F}^3 \neq U_1 \oplus U_2 \oplus U_3\) since, for example,

\[\{(0, 0, 0) = (0, 1, 0) + (0, 0, 1) + (0, −1, −1)\}\]
But $\cap(U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \emptyset)$ so that the analog of Proposition 4.4.7 does not hold.

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**Contributors**

- Isaiah Lankham, Mathematics Department at UC Davis
- Bruno Nachtergaele, Mathematics Department at UC Davis
- Anne Schilling, Mathematics Department at UC Davis

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