4.4: Sums and direct sum

Throughout this section, \( V \) is a vector space over \( \mathbb{F} \), and \( U_1, U_2 \subset V \) denote subspaces.

Definition 4.4.1: (subspace) sum

Let \( U_1, U_2 \subset V \) be subspaces of \( V \). Define the **(subspace) sum** of \( U_1 \) and \( U_2 \) to be the set

and \( U_2 \) to be the set

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**Figure 4.4.1**: The union \( (U \cup U') \) of two subspaces is not necessarily a subspace.
\[ U_1 + U_2 = \{ u_1 + u_2 | u_1 \in U_1, u_2 \in U_2 \}. \tag{4.4.1} \]

Check as an exercise that \((U_1 + U_2)\) is a subspace of \((V)\). In fact, \((U_1 + U_2)\) is the smallest subspace of \((V)\) that contains both \((U_1)\) and \((U_2)\).

**Example 4.4.2.** Let
\[ U_1 = \{(x, 0, 0) \in \mathbb{F}^3 | x \in \mathbb{F}\}, \]
\[ U_2 = \{(0, y, 0) \in \mathbb{F}^3 | y \in \mathbb{F}\}. \]
Then
\[ U_1 + U_2 = \{(x, y, 0) \in \mathbb{F}^3 | x, y \in \mathbb{F}\}. \tag{4.4.2} \]

If, alternatively, \((U_2 = \{(y, y, 0) \in \mathbb{F}^3 | y \in \mathbb{F}\})\), then Equation (4.4.2) still holds.

If \((U = U_1 + U_2)\), then for any \((u \in U)\), there exist \((u_1 \in U_1)\) and \((u_2 \in U_2)\) such that \((u = u_1 + u_2)\).

If it so happens that \((u)\) can be uniquely written as \((u_1 + u_2)\), then \((U)\) is called the **direct sum** of \((U_1)\) and \((U_2)\).

**Definition 4.4.3: Direct Sum**

Suppose every \((u \in U)\) can be uniquely written as \((u = u_1 + u_2)\) for \((u_1 \in U_1)\) and \((u_2 \in U_2)\). Then we use
\[ U = U_1 \oplus U_2\]
to denote the **direct sum** of \((U_1)\) and \((U_2)\).

**Example 4.4.4.** Let
\[ U_1 = \{(x, y, 0) \in \mathbb{F}^3 | x, y \in \mathbb{F}\}, \]
\[ U_2 = \{(0, 0, z) \in \mathbb{F}^3 | z \in \mathbb{F}\}. \]
Then \((\mathbb{F}^3 = U_1 \oplus U_2)\). However, if instead
\[ U_2 = \{(0, w, z) | w, z \in \mathbb{F}\}, \]
then \((\mathbb{F}^3 = U_1 + U_2)\) but is not the direct sum of \((U_1)\) and \((U_2)\).

**Example 4.4.5.** Let
\[ U_1 = \{ p \in \mathbb{F}[z] | p(z) = a_0 + a_2 z^2 + \cdots + a_{2m+1}z^{m+1} \}, \]
\[ U_2 = \{ p \in \mathbb{F}[z] | p(z) = a_1z + a_3z^3 + \cdots + a_{2m+1}z^{2m+1} \}. \]
Then \((\mathbb{F}[z] = U_1 \oplus U_2)\).

**Proposition 4.4.6.** Let \((U_1, U_2 \subset V)\) be subspaces. Then \((V = U_1 \oplus U_2)\) if and only if the following two conditions hold:

1. \((V = U_1 + U_2)\)
2. If \((0 = u_1 + u_2)\) with \((u_1 \in U_1)\) and \((u_2 \in U_2)\), then \((u_1 = u_2 = 0)\).
Proof.

\(\Rightarrow\) Suppose \(V = U_1 \oplus U_2\). Then Condition 1 holds by definition. Certainly \(0 = 0 + 0\), and, since by uniqueness this is the only way to write \(0 \in V\), we have \(u_1 = u_2 = 0\).

\(\Leftarrow\) Suppose Conditions 1 and 2 hold. By Condition 1, we have that, for all \(v \in V\), there exist \(u_1 \in U_1\) and \(u_2 \in U_2\) such that \(v = u_1 + u_2\). Suppose \(v = w_1 + w_2\) with \(w_1 \in U_1\) and \(w_2 \in U_2\). Subtracting the two equations, we obtain
\[
0 = (u_1 - w_1) + (u_2 - w_2),
\]
where \((u_1 - w_1) \in U_1\) and \((u_2 - w_2) \in U_2\). By Condition 2, this implies \((u_1 - w_1 = 0)\) and \((u_2 - w_2 = 0)\), or equivalently \((u_1 = w_1)\) and \((u_2 = w_2)\), as desired.

**Proposition 4.4.7.** Let \((U_1, U_2) \subset V\) be subspaces. Then \(V = U_1 \oplus U_2\) if and only if the following two conditions hold:

1. \(V = U_1 + U_2;\)
2. \(U_1 \cap U_2 = \{0\}\)

Proof.

\(\Rightarrow\) Suppose \(V = U_1 \oplus U_2\). Then Condition 1 holds by definition. If \(u \in U_1 \cap U_2\), then \(0 = u + (−u)\) with \(u \in U_1\) and \(−u \in U_2\) (why?). By Proposition 4.4.6, we have \(u = 0\) and \(−u = 0\) so that \(U_1 \cap U_2 = \{0\}\).

\(\Leftarrow\) Suppose Conditions 1 and 2 hold. To prove that \(V = U_1 \oplus U_2\) holds, suppose that
\[
0 = u_1 + u_2, \text{ where } u_1 \in U_1 \text{ and } u_2 \in U_2. \tag{4.3}
\]
By Proposition 4.4.6, it succed to show that \(u_1 = u_2 = 0\). Equation (4.3) implies that \(u_1 = −u_2 \in U_2\). Hence \(u_1 \in U_1 \cap U_2\), which in turn implies that \(u_1 = 0\). It then follows that \(u_2 = 0\) as well.

Everything in this section can be generalized to \(m\) subspaces \((U_1, U_2, \ldots U_m)\) with the notable exception of Proposition 4.4.7. To see, this consider the following example.

**Example 4.4.8.** Let
\[
U_1 = \{(x, y, 0) \in \mathbb{F}^3 | x, y \in \mathbb{F}\},
U_2 = \{(0, 0, z) \in \mathbb{F}^3 | z \in \mathbb{F}\},
U_3 = \{(0, y, y) \in \mathbb{F}^3 | y \in \mathbb{F}\}.
\]
Then certainly \((\mathbb{F}^3 = U_1 + U_2 + U_3)\), but \((\mathbb{F}^3 \neq U_1 \oplus U_2 \oplus U_3)\) since, for example,
\[
(0, 0, 0) = (0, 1, 0) + (0, 0, 1) + (0, −1, −1).
\]
But \( U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{0\} \) so that the analog of Proposition 4.4.7 does not hold.

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