1.3: Applications and Solving Right Triangles

Throughout its early development, trigonometry was often used as a means of indirect measurement, e.g. determining large distances or lengths by using measurements of angles and small, known distances. Today, trigonometry is widely used in physics, astronomy, engineering, navigation, surveying, and various fields of mathematics and other disciplines. In this section we will see some of the ways in which trigonometry can be applied. Your calculator should be in degree mode for these examples.

Example 1.11

A person stands \((150\text{ ft})\) away from a flagpole and measures an angle of elevation of \((32^\circ)\) from his horizontal line of sight to the top of the flagpole. Assume that the person's eyes are a vertical distance of 6 ft from the ground. What is the height of the flagpole?

Solution:
The picture on the right describes the situation. We see that the height of the flagpole is \((h + 6)\) ft, where

\[
\frac{h}{150} \approx \tan 32^\circ \Rightarrow h \approx 150 \cdot \tan 32^\circ \\
\approx 150 \cdot (0.6249) \approx 94 ~.\]

How did we know that \(\tan 32^\circ = 0.6249\)? By using a calculator. And since none of the numbers we were given had decimal places, we rounded off the answer for \(h\) to the nearest integer. Thus, the height of the flagpole is \(h + 6 = 94 + 6 = \boxed{100 \text{ ft}}\).

Example 1.12

A person standing \(400\) ft from the base of a mountain measures the angle of elevation from the ground to the top of the mountain to be \(25^\circ\). The person then walks \(500\) ft straight back and measures the angle of elevation to now be \(20^\circ\). How tall is the mountain?

Solution:

We will assume that the ground is flat and not inclined relative to the base of the mountain. Let \(h\) be the height of the mountain, and let \(x\) be the distance from the base of the mountain to the point directly beneath the top of the mountain, as in the picture on the right. Then we see that

\[
\begin{align}
\frac{h}{x + 400} &= \tan 25^\circ \quad \Rightarrow \quad h = (x + 400) \cdot \tan 25^\circ \\
\frac{h}{x + 400 + 500} &= \tan 20^\circ \quad \Rightarrow \quad h = (x + 900) \cdot \tan 20^\circ \\
\end{align}
\]

\((x + 400) \cdot \tan 25^\circ = (x + 900) \cdot \tan 20^\circ\), since they both equal \(h\). Use that equation to solve for \(x\):

\[
\begin{align}
x \cdot \tan 25^\circ &= x \cdot \tan 20^\circ \\
900 \cdot \tan 20^\circ &= 400 \cdot \tan 25^\circ
\end{align}
\]
Example 1.13

A blimp \(4280\) ft above the ground measures an angle of depression of \(24^\circ\) from its horizontal line of sight to the base of a house on the ground. Assuming the ground is flat, how far away along the ground is the house from the blimp?

Solution:

Let \(x\) be the distance along the ground from the blimp to the house, as in the picture to the right. Since the ground and the blimp's horizontal line of sight are parallel, we know from elementary geometry that the angle of elevation \(\theta\) from the base of the house to the blimp is equal to the angle of depression from the blimp to the base of the house, i.e. \(\theta = 24^\circ\). Hence,

\[
\frac{4280}{x} \approx \tan{24^\circ} \quad \Rightarrow \quad x \approx \frac{4280}{\tan{24^\circ}} \approx \boxed{9613 \text{ ft}}.
\]

Example 1.14

An observer at the top of a mountain \(3\) miles above sea level measures an angle of depression of \(2.23^\circ\) to the ocean horizon. Use this to estimate the radius of the earth.
Solution:

We will assume that the earth is a sphere. Let \( r \) be the radius of the earth. Let the point \( A \) represent the top of the mountain, and let \( H \) be the ocean horizon in the line of sight from \( A \), as in Figure 1.3.1. Let \( O \) be the center of the earth, and let \( B \) be a point on the horizontal line of sight from \( A \) (i.e. on the line perpendicular to \( \overline{OA} \)). Let \( \theta \) be the angle \( \angle AOH \).

Since \( A \) is 3 miles above sea level, we have \( OA = r + 3 \). Also, \( OH = r \). Now since \( \overline{AB} \perp \overline{OA} \), we have \( \angle OAH = 90^\circ \), so we see that \( \cos \theta = 0.9999 \). We see that the line through \( A \) and \( H \) is a tangent line to the surface of the earth (considering the surface as the circle of radius \( r \) through \( H \) as in the picture). So by Exercise 14 in Section 1.1, \( \angle OH \perp \overline{OA} \) and hence \( \angle OHA = 90^\circ \). Since the angles in the triangle \( \triangle OAH \) add up to \( 180^\circ \), we have \( \theta = 180^\circ - 90^\circ - 87.77^\circ = 2.23^\circ \). Thus,

\[
\cos \theta = \frac{OH}{OA} = \frac{r}{r + 3} \quad \Rightarrow \quad \frac{r}{r + 3} = \cos 2.23^\circ ,
\]

so solving for \( r \) we get

\[
\begin{align}
    r &= (r + 3) \cos 2.23^\circ \\
    r &= r \cos 2.23^\circ = 3 \cos 2.23^\circ \quad \Rightarrow \quad r = \frac{3 \cos 2.23^\circ}{1 - \cos 2.23^\circ} \\
    r &= 3958.3 \text{ miles}
\end{align}
\]
Note: This answer is very close to the earth's actual (mean) radius of \(3956.6\) miles.

Example 1.15

As another application of trigonometry to astronomy, we will find the distance from the earth to the sun. Let \(\text{O}\) be the center of the earth, let \(\text{A}\) be a point on the equator, and let \(\text{B}\) represent an object (e.g. a star) in space, as in the picture on the right. If the earth is positioned in such a way that the angle \(\angle OAB = 90^\circ\), then we say that the angle \(\alpha = \angle OBA\) is the *equatorial parallax* of the object. The equatorial parallax of the sun has been observed to be approximately \(\alpha \approx 0.00244^\circ\). Use this to estimate the distance from the center of the earth to the sun.

![Diagram of Earth, Sun, and Object](image)

**Solution:**

Let \(\text{B}\) be the position of the sun. We want to find the length of \(\overline{OB}\). We will use the actual radius of the earth, mentioned at the end of Example 1.14, to get \(\text{OA} = 3956.6\) miles. Since \(\angle OAB = 90^\circ\), we have

\[
\frac{\text{OA}}{\text{OB}} = \sin \alpha \quad \Rightarrow \quad \text{OB} = \frac{\text{OA}}{\sin \alpha} = \frac{3956.6}{\sin 0.00244^\circ} \approx 92908394
\]

so the distance from the center of the earth to the sun is approximately \(\fbox{93\text{ million miles}}\).

Note: The earth's orbit around the sun is an ellipse, so the actual distance to the sun varies.

In the above example we used a very small angle \(0.00244^\circ\). A degree can be divided into smaller units: a *minute* is one-sixtieth of a degree, and a *second* is one-sixtieth of a minute. The symbol for a minute is \('\) and the symbol for a second is \(''\). For example, \(4.505^\circ = 4^\circ 30' 18''\). And \(4^\circ 30'; 30'' = 4^\circ 30' 18''\):

\[4^\circ 30'; 30'' \approx 4 + \frac{30}{60} + \frac{18}{3600} \text{ degrees} \approx 4.505^\circ\]

In Example 1.15 we used \(\alpha = 0.00244^\circ \approx 8.8''\), which we mention only because some angle measurement...
An observer on earth measures an angle of \(32'4''\) from one visible edge of the sun to the other (opposite) edge, as in the picture on the right. Use this to estimate the radius of the sun.

**Solution:**

Let the point \(E\) be the earth and let \(S\) be the center of the sun. The observer's lines of sight to the visible edges of the sun are tangent lines to the sun's surface at the points \(A\) and \(B\). Thus, \(\angle EAS = \angle EBS = 90^\circ\). The radius of the sun equals \(AS\). Clearly \(AS = BS\). So since \(EB = EA\) (why?), the triangles \(\triangle EAS\) and \(\triangle EBS\) are similar. Thus, \(\angle AES = \angle BES = \frac{1}{2} \angle AEB = \frac{1}{2} (32'4'') = 16'2'' = (16/60) = (2/3600) = 0.26722^\circ\).\)

Now, \(ES\) is the distance from the surface of the earth (where the observer stands) to the center of the sun. In Example 1.15 we found the distance from the center of the earth to the sun to be \(92,908,394\) miles. Since we treated the sun in that example as a point, then we are justified in treating that distance as the distance between the centers of the earth and sun. So \(ES = 92908394 - \text{radius of earth} = 92908394 - 3956.6 = 92904437.4\) miles. Hence,

\[
\sin(\angle AES) \approx \frac{AS}{ES} \quad \Rightarrow \quad AS \approx ES \sin(0.26722^\circ) = (92904437.4) \sin(0.26722^\circ) \approx 433,293 \text{ miles}\.
\]

Note: This answer is close to the sun's actual (mean) radius of \(432,200\) miles.

You may have noticed that the solutions to the examples we have shown required at least one right triangle. In applied problems it is not always obvious which right triangle to use, which is why these sorts of problems can be difficult. Often no right triangle will be immediately evident, so you will have to create one. There is no general strategy for this, but remember that a right triangle requires a right angle, so look for places where you can form perpendicular line segments. When the problem contains a circle, you can create right angles by using the perpendicularity of the tangent line to the circle at a point with the line that joins that point to the center of the circle. We did exactly that in Examples 1.14, 1.15, and 1.16.

Example 1.17
The machine tool diagram on the right shows a symmetric *V-block*, in which one circular roller sits on top of a smaller circular roller. Each roller touches both slanted sides of the V-block. Find the diameter \(d\) of the large roller, given the information in the diagram.

![Diagram of V-block with labeled angles and dimensions]

**Solution:**

The diameter \(d\) of the large roller is twice the radius \(OB\), so we need to find \(OB\). To do this, we will show that \(\triangle OBC\) is a right triangle, then find the angle \(\angle BOC\), and then find \(BC\). The length \(OB\) will then be simple to determine.

Since the slanted sides are tangent to each roller, \(\angle ODA = \angle PEC = 90^\circ\). By symmetry, since the vertical line through the centers of the rollers makes a \(37^\circ\) angle with each slanted side, we have \(\angle OAD = 37^\circ\). Hence, since \(\triangle ODA\) is a right triangle, \(\angle DOA\) is the complement of \(\angle OAD\). So \(\angle DOA = 53^\circ\).

Since the horizontal line segment \(BC\) is tangent to each roller, \(\angle OBC = \angle PBC = 90^\circ\). Thus, \(\triangle OBC\) is a right triangle. And since \(\angle ODA = 90^\circ\), we know that \(\triangle ODC\) is a right triangle. Now, \(\triangle OBC\) (since they each equal the radius of the large roller), so by the Pythagorean Theorem we have \(BC = DC\):

\[
BC^2 = OC^2 - OB^2 = BC^2 = DC^2 \quad \Rightarrow \quad BC = DC
\]

Thus, \(\triangle OBC\) and \(\triangle ODC\) are congruent triangles (which we denote by \(\triangle OBC \cong \triangle ODC\)), since their corresponding sides are equal. Thus, their corresponding angles are equal. So in particular, \(\angle BOC = \angle DOC\). We know that \(\angle DOB = \angle DOA = 53^\circ\). Thus,

\[
\angle DOB = \angle BOC = \angle DOC = \angle BOC + \angle DOC = \angle BOC
\]
\[2\angle\,BOC \quad \Rightarrow \quad \angle\,BOC \approx 26.5^\circ.\]

Likewise, since \(\angle\,BPC = \angle\,PEC = 90^\circ\) and \(\triangle\,BPC\) and \(\triangle\,EPC\) are congruent right triangles. Thus, \(\angle\,(BC = EC)\). But we know that \(\angle\,(BC = DC)\), and we see from the diagram that \(\angle\,(EC + DC = 1.38)\). Thus, \(\angle\,(BC + BC = 1.38)\) and so \(\angle\,(BC = 0.69)\). We now have all we need to find \(\angle\,(OB)\):

\[
\frac{BC}{OB} \approx \tan\,\angle\,BOC \quad \Rightarrow \quad OB \approx \frac{BC}{\tan\,\angle\,BOC} \approx \frac{0.69}{\tan\,26.5^\circ} \approx 1.384\]

Hence, the diameter of the large roller is \(d = 2\times OB = 2\times(1.384) = \boxed{2.768}\).

Example 1.18

A slider-crank mechanism is shown in Figure 1.3.2 below. As the piston moves downward the connecting rod rotates the crank in the clockwise direction, as indicated.

![Slider-crank mechanism](image)

Figure 1.3.2 Slider-crank mechanism

The point \(A\) is the center of the connecting rod's wrist pin and only moves vertically. The point \(B\) is the center of the crank pin and moves around a circle of radius \(r\) centered at the point \(O\), which is directly below \(A\) and does not move. As the crank rotates it makes an angle \(\theta\) with the line \(\overline{OA}\). The instantaneous center of rotation of the
connecting rod at a given time is the point \( C \) where the horizontal line through \( A \) intersects the extended line through \( O \) and \( B \). From Figure 1.3.2 we see that \( \angle OAC = 90^\circ \), and we let \( a = AC \), \( b = AB \), and \( c = BC \).

In the exercises you will show that for \( 0^\circ < \theta < 90^\circ \),

\[
\begin{align*}
\text{c} &= \frac{\sqrt{b^2 - r^2 \sin^2 \theta}}{\cos \theta} \quad \text{and} \quad \text{a} = r \sin \theta + \sqrt{b^2 - r^2 \sin^2 \theta} \tan \theta.
\end{align*}
\]

For some problems it may help to remember that when a right triangle has a hypotenuse of length \( r \) and an acute angle \( \theta \), as in the picture below, the adjacent side will have length \( r \cos \theta \) and the opposite side will have length \( r \sin \theta \). You can think of those lengths as the horizontal and vertical "components" of the hypotenuse.

\[
\begin{align*}
\text{r} &\quad \sin \theta \\
\text{r} &\quad \cos \theta
\end{align*}
\]

Notice in the above right triangle that we were given two pieces of information: one of the acute angles and the length of the hypotenuse. From this we determined the lengths of the other two sides, and the other acute angle is just the complement of the known acute angle. In general, a triangle has six parts: three sides and three angles. **Solving a triangle** means finding the unknown parts based on the known parts. In the case of a right triangle, one part is always known: one of the angles is \( 90^\circ \).

**Example 1.19**

Solve the right triangle in Figure 1.3.3 using the given information:

\[
\begin{align*}
\text{(a)} \quad &c = 10, \quad A = 22^\circ \\
\text{Solution:} &\quad \text{The unknown parts are } a, b, \text{ and } B. \text{ Solving yields:}
\end{align*}
\]

\[
\begin{align*}
a &\approx c \sin A = 10 \sin 22^\circ \approx 3.75 \\
b &\approx c \cos A = 10 \cos 22^\circ \approx 9.27 \\
B &\approx 90^\circ - A = 90^\circ - 22^\circ \approx 68^\circ
\end{align*}
\]
(b) \(b = 8, A = 40^\circ\)

**Solution**: The unknown parts are \(a\), \(c\), and \(B\). Solving yields:

\[
\begin{align}
\frac{a}{b} &= \tan A \quad \Rightarrow \quad a &= b \tan A = 8 \tan 40^\circ = 6.71 \\
\frac{b}{c} &= \cos A \quad \Rightarrow \quad c &= \frac{b}{\cos A} = \frac{8}{\cos 40^\circ} = 10.44
\end{align}
\]

(c) \(a = 3, b = 4\)

**Solution**: The unknown parts are \(c\), \(A\), and \(B\). By the Pythagorean Theorem,

\[c = \sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5\]

Now, \(\tan A = \frac{a}{b} = \frac{3}{4} = 0.75\). So how do we find \(A\)? There should be a key labeled \(\tan^{-1}\) on your calculator, which works like this: give it a number \(x\) and it will tell you the angle \(\theta\) such that \(\tan \theta = x\). In our case, we want the angle \(A\) such that \(\tan A = 0.75\):

\[
\text{Enter: } 0.75 \quad \text{Press: } \tan^{-1} \quad \text{Answer: } 36.86989765
\]

This tells us that \(A = 36.87^\circ\), approximately. Thus \(B = 90^\circ - A = 90^\circ - 36.87^\circ = 53.13^\circ\).

Note: The \(\sin^{-1}\) and \(\cos^{-1}\) keys work similarly for sine and cosine, respectively. These keys use the *inverse trigonometric functions*, which we will discuss in Chapter 5.

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