7.E: Elliptic Equations of Second Order (Exercises)

These are homework exercises to accompany Miersemann's "Partial Differential Equations" Textmap. This is a textbook targeted for a one semester first course on differential equations, aimed at engineering students. Partial differential equations are differential equations that contains unknown multivariable functions and their partial derivatives. Prerequisite for the course is the basic calculus sequence.

Q7.1

Let \( \gamma(x,y) \) be a fundamental solution to \( \triangle \), \( y \in \Omega \). Show that
\[
-\int_\Omega \gamma(x,y) \ \triangle \Phi(x) \ dx = \Phi(y) \quad \text{for all} \quad \Phi \in C_0^2(\Omega) .
\]

\textbf{Hint:} See the proof of the representation formula.

Q7.2

Show that \( |x|^{-1} \sin(k|x|) \) is a solution of the Helmholtz equation
\[
\triangle u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \{0\} .
\]
Q7.3

Assume \( u \in C^2(\overline{\Omega}) \), \( \Omega \) bounded and sufficiently regular, is a solution of
\[
\begin{align*}
\triangle u &= u^3 \quad \text{in} \quad \Omega \\
u &= 0 \quad \text{on} \quad \partial \Omega.
\end{align*}
\]
Show that \( u = 0 \) in \( \Omega \).

Q7.4

Let \( \Omega_\alpha = \{ x \in \mathbb{R}^2 : x_1 > 0, 0 < x_2 < x_1 \tan \alpha \} \), \( 0 < \alpha \le \pi \). Show that
\[
u(x) = r^{\pi/\alpha k} \sin \left( \frac{\pi}{\alpha} k \theta \right)
\]
is a harmonic function in \( \Omega_\alpha \) satisfying \( u = 0 \) on \( \partial \Omega_\alpha \), provided \( k \) is an integer. Here \((r, \theta)\) are polar coordinates with the center at \((0,0)\).

Q7.5

Let \( u \in C^2(\overline{\Omega}) \) be a solution of \( \triangle u = 0 \) on the quadrangle \( \Omega = (0,1) \times (0,1) \) satisfying the boundary conditions \( u(0,y) = u(1,y) = 0 \) for all \( y \in [0,1] \) and \( u_y(x,0) = u_y(x,1) = 0 \) for all \( x \in [0,1] \). Prove that \( u \equiv 0 \) in \( \overline{\Omega} \).

Q7.6

Let \( u \in C^2(\mathbb{R}^n) \) be a solution of \( \triangle u = 0 \) in \( \mathbb{R}^n \) satisfying \( u \in L^2(\mathbb{R}^n) \), i.e., \( \int_{\mathbb{R}^n} u^2(x) \, dx < \infty \). Show that \( u \equiv 0 \) in \( \mathbb{R}^n \).

**Hint:** Prove
\[
\int_{B_R(0)} |\nabla u|^2 \, dx \leq \text{const.} \int_{B_{2R}(0)} |u|^2 \, dx,
\]
where \( c \) is a constant independent of \( R \).

To show this inequality, multiply the differential equation by \( \inteta := \eta^2 u \), where
\( \eta \in C^1 \) is a cut-off function with properties: \( \eta \equiv 1 \) in \( B_R(0) \), \( \eta \equiv 0 \) in the exterior of \( B_{2R}(0) \), \( \inteta \equiv 1 \), \( |\nabla \eta| \leq C/R \). Integrate the product, apply integration by parts and use
the formula \(2ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2\), \((\epsilon > 0)\).

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**Q7.7**

Show that a bounded harmonic function defined on \(\mathbb{R}^n\) must be a constant (a theorem of Liouville).

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**Q7.8**

Assume \(u \in C^2(B_1(0)) \cap C(\overline{B_1(0)} \setminus \{(1,0)\})\) is a solution of

\[
\begin{align*}
\triangle u &= 0 \quad \text{in} \quad B_1(0) \\
u &= 0 \quad \text{on} \quad \partial B_1(0) \setminus \{(1,0)\}.
\end{align*}
\]

Show that there are at least two solutions.

*Hint:* Consider

\[
u(x,y) = \frac{1-(x^2+y^2)}{(1-x)^2+y^2}.
\]

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**Q7.9**

Assume \(\Omega \subset \mathbb{R}^n\) is bounded and \((u,v) \in C^2(\Omega) \cap C(\overline{\Omega})\) satisfy

\(\triangle u = \triangle v\) and \(\max_{\partial \Omega} |u-v| \leq \epsilon\) for given \((\epsilon > 0)\). Show that

\(\max_{\overline{\Omega}} |u-v| \leq \epsilon\).

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**Q7.10**

Set \(\Omega = \mathbb{R}^n \setminus \overline{B_1(0)}\) and let \((u,v) \in C^2(\overline{\Omega})\) be a harmonic function in \(\Omega\) satisfying \(\lim_{|x| \to \infty} u(x) = 0\). Prove that

\[
\max_{\overline{\Omega}} |u| = \max_{\partial \Omega} |u|.
\]

*Hint:* Apply the maximum principle to \(\Omega \cap B_R(0)\), \(R\) large.
Q7.11

Let \( \Omega_{\alpha} = \{ x \in \mathbb{R}^2 : x_1 > 0, 0 < x_2 < x_1 \tan \alpha \} \), \( 0 < \alpha \leq \pi \), \( \Omega_{\alpha,R} = \Omega_{\alpha} \cap B_R(0) \), and assume \( f \) is given and bounded on \( \overline{\Omega_{\alpha,R}} \).

Show that for each solution \( u \in C^1(\overline{\Omega_{\alpha,R}}) \cap C^2(\Omega_{\alpha,R}) \) of \( \triangle u = f \) in \( \Omega_{\alpha,R} \) satisfying \( u = 0 \) on \( \partial \Omega_{\alpha,R} \cap B_R(0) \), holds:

For given \( \epsilon > 0 \) there is a constant \( C(\epsilon) \) such that

\[
|u(x)| \leq C(\epsilon) |x|^{{\pi/\alpha} - \epsilon}
\]

in \( \Omega_{\alpha,R} \).

**Hint:** (a) Comparison principle (a consequence from the maximum principle): Assume \( \Omega \) is bounded, \( u,v \in C^2(\overline{\Omega}) \cap C(\overline{\Omega}) \) satisfying \( -\triangle u \leq -\triangle v \) in \( \Omega \) and \( u \leq v \) on \( \partial \Omega \). Then \( u \leq v \) in \( \Omega \).

(b) An appropriate comparison function is

\[
v = A r^{{\pi/\alpha} - \epsilon} \sin(B(\theta + \eta)) ,
\]

\( A, B, \eta \) appropriate constants, \( B, \eta \) positive.

Q7.12

Let \( \Omega \) be the quadrangle \( (-1,1) \times (-1,1) \) and \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) a solution of the boundary value problem \( -\triangle u = 1 \) in \( \Omega \), \( u = 0 \) on \( \partial \Omega \). Find a lower and an upper bound for \( u(0,0) \).

**Hint:** Consider the comparison function \( v = A(x^2 + y^2) \), \( A = \text{const.} \).

Q7.13

Let \( u \in C^2(B_a(0)) \cap C(\overline{B_a(0)}) \) satisfying \( u \geq 0, \triangle u = 0 \) in \( B_a(0) \). Prove (Harnack's inequality):

\[
\frac{a^{n-2}(a-|\zeta|)}{(a+|\zeta|)^{n-1}} u(0) \leq u(\zeta) \leq \frac{a^{n-2}(a+|\zeta|)}{(a-|\zeta|)^{n-1}} u(0) .
\]
Hint: Use the formula (see Theorem 7.2)

\[
\begin{align*}
  u(y) &= \frac{a^2 - |y|^2}{a\omega_n} \int_{|x|=a} \frac{u(x)}{|x-y|^n} \, dS_x \\
  &\quad \text{for } (y = \zeta) \text{ and } (y = 0).
\end{align*}
\]

Q7.14

Let \( \phi(\theta) \) be a \((2\pi)-\)periodic \(C^4\)-function with the Fourier series

\[
\phi(\theta) = \sum_{n=0}^{\infty} \left( a_n \cos(n\theta) + b_n \sin(n\theta) \right)
\]

Show that

\[
\sum_{n=0}^{\infty} \left( a_n \cos(n\theta) + b_n \sin(n\theta) \right) r^n
\]

solves the Dirichlet problem in \(B_1(0)\).

Q7.15

Assume \(u \in C^2(\Omega)\) satisfies \(\triangle u = 0\) in \(\Omega\). Let \(B_a(\zeta)\) be a ball such that its closure is in \(\Omega\).

Show that

\[
|D^{\alpha} u(\zeta)| \leq M \left( \frac{||\alpha||\gamma_n}{a} \right)^{||\alpha||},
\]

where \(M = \sup_{x \in B_a(\zeta)} |u(x)|\) and \(\gamma_n = 2n\omega_{n-1}/((n-1)\omega_n)\).

Hint: Use the formula of Theorem 7.2, successively to the \(k\)th derivatives in balls with radius \(a(||\alpha||-k)/m\), \(k=0,1,\ldots,m-1\).

Q7.16

Use the result of the previous exercise to show that \(u \in C^2(\Omega)\) satisfying \(\triangle u = 0\) in \(\Omega\) is real analytic in \(\Omega\).

Hint: Use Stirling’s formula
\(n!=n^ne^{-n}\sqrt{2\pi n}+O\left(\frac{1}{\sqrt{n}}\right)\) as \(n\to\infty\), to show that \((u)\) is in the class \(C_{K,r}(\zeta)\), where \(K=cM\) and \(r=a/(\gamma_n)\). The constant \(c\) is the constant in the estimate \(n^n\leq ce^n\) which follows from Stirling's formula. See Section 3.5 for the definition of a real analytic function.

**Q7.17**

Assume \((\Omega)\) is connected and \((u\in C^2(\Omega))\) is a solution of \(\Delta u=0\) in \((\Omega)\). Prove that \((u\equiv0)\) in \((\Omega)\) if \((D^\alpha u(\zeta)=0)\) for all \((\alpha)\), for a point \((\zeta\in\Omega)\). In particular, \((u\equiv0)\) in \((\Omega)\) if \((u\equiv0)\) in an open subset of \((\Omega)\).

**Q7.18**

Let \((\Omega=((x_1,x_2,x_3)\in\mathbb{R}^3:x_3>0))\), which is a half-space of \((\mathbb{R}^3)\). Show that
\[
G(x,y)=\frac{1}{4\pi|x-y|}-\frac{1}{4\pi|x-\overline{y}|},
\]
where \((\overline{y}=(y_1,y_2,-y_3))\), is the Green function to \((\Omega)\).

**Q7.19**

Let \((\Omega=((x_1,x_2,x_3)\in\mathbb{R}^3:x_1^2+x_2^2+x_3^2<R^2,x_3>0))\), which is half of a ball in \((\mathbb{R}^3)\). Show that
\[
\begin{align*}
G(x,y)&=\frac{1}{4\pi|x-y|}-\frac{R}{4\pi|y||x-y'|}+
\frac{1}{4\pi|x-\overline{y}'|},
\end{align*}
\]
where \((\overline{y}=(y_1,y_2,-y_3))\), \((y'=R^2y/(|y|^2))\) and \((\overline{y}'=R^2\overline{y}/(|y|^2))\), is the Green function to \((\Omega)\).

**Q7.20**

Let \((\Omega=((x_1,x_2,x_3)\in\mathbb{R}^3:x_2>0,x_3>0))\), which is a wedge in \((\mathbb{R}^3)\). Show that
\[
\begin{align*}
G(x,y)&=\frac{1}{4\pi|x-y|}-\frac{1}{4\pi|x-\overline{y}|}+
\frac{1}{4\pi|x-y'|}+
\frac{1}{4\pi|x-\overline{y}'|},
\end{align*}
\]
where \((\overline{y}=(y_1,y_2,-y_3))\), \((y'=(y_1,-y_2,y_3))\) and \((\overline{y}'=(y_1,-y_2,-y_3))\),
is the Green function to \( \Omega \).

**Q7.21**

Find Green's function for the exterior of a disk, i.e., of the domain \( \Omega = \{x \in \mathbb{R}^2 : |x| > R \} \).

**Q7.22**

Find Green's function for the angle domain \( \Omega = \{z \in \mathbb{C} : 0 < \arg z < \alpha \pi \} \), \( 0 < \alpha < \pi \).

**Q7.23**

Find Green's function for the slit domain \( \Omega = \{z \in \mathbb{C} : 0 < \arg z < 2\pi \} \).

**Q7.24**

Let for a sufficiently regular domain \( \Omega \in \mathbb{R}^n \), a ball or a quadrangle for example,

\[
F(x) = \int_{\Omega} K(x,y) \, dy,
\]

where \( K(x,y) \) is continuous in \( \overline{\Omega} \times \overline{\Omega} \) where \( x \neq y \), and which satisfies

\[
|K(x,y)| \leq \frac{c}{|x-y|^{\alpha}}
\]

with a constants \( c \) and \( \alpha, (\alpha < n) \).

Show that \( F(x) \) is continuous on \( \overline{\Omega} \).

**Q7.25**

Prove (i) of the lemma of Section 7.5.

*Hint:* Consider the case \( n \geq 3 \). Fix a function \( \eta \in C^1(\mathbb{R}) \) satisfying \( \eta(t) \leq 1 \), \( \eta(t) = 1 \) for \( |t| \leq 1 \), \( \eta(t) = 0 \) for \( |t| \geq 2 \) and consider for \( \eta' > 0 \) the regularized integral

\[
V_{\epsilon}(x) = \int_{\Omega} f(y) \eta_{\epsilon} \frac{dy}{|x-y|^{n-2}},
\]

where \( \eta_{\epsilon} = \eta(|x-y|/\epsilon) \). Show that \( V_{\epsilon} \) converges uniformly to \( V \) on compact subsets of \( \mathbb{R}^n \) as \( \epsilon \to 0 \), and that \( \eta(x) V_{\epsilon}(x) \) converges uniformly on compact subsets of...
\(\mathbb{R}^n\) to

$$\int_{\Omega} f(y) \frac{\partial}{\partial x_i} \left( \frac{1}{|x-y|^{n-2}} \right) \, dy$$
as \(\epsilon \to 0\).

\section*{Q7.26}

Consider the inhomogeneous Dirichlet problem \(-\triangle u = f\) in \(\Omega\), \(u = \phi\) on \(\partial\Omega\). Transform this problem into a Dirichlet problem for the Laplace equation.

\textit{Hint:} Set \(u = w + v\), where \(w(x) = \int_{\Omega} s(|x-y|) f(y) \, dy\).

\section*{Contributors}

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