3.4: Calculating Power- Banzhaf Power Index

The Banzhaf power index was originally created in 1946 by Lionel Penrose, but was reintroduced by John Banzhaf in 1965. The power index is a numerical way of looking at power in a weighted voting situation.

Calculating Banzhaf Power Index

To calculate the Banzhaf power index:

1. List all winning coalitions
2. In each coalition, identify the players who are critical
3. Count up how many times each player is critical
4. Convert these counts to fractions or decimals by dividing by the total times any player is critical

Example 4

Find the Banzhaf power index for the voting system \([8: 6, 3, 2]\).

Solution

We start by listing all winning coalitions. If you aren’t sure how to do this, you can list all coalitions, then eliminate the non-winning coalitions. No player is a dictator, so we’ll only consider two and three player coalitions.

\(\{P_1, P_2\}\) Total weight: 9. Meets quota.

\(\{P_1, P_3\}\) Total weight: 8. Meets quota.
\(\{P_2, P_3\}\) Total weight: 5. Does not meet quota.

\(\{P_1, P_2, P_3\}\) Total weight: 11. Meets quota.

Next we determine which players are critical in each winning coalition. In the winning two-player coalitions, both players are critical since no player can meet quota alone. Underlining the critical players to make it easier to count:

\(\{\underline{P}_1, \underline{P}_2\}\)
\(\{\underline{P}_1, \underline{P}_3\}\)

In the three-person coalition, either \(P_2\) or \(P_3\) could leave the coalition and the remaining players could still meet quota, so neither is critical. If \(P_1\) were to leave, the remaining players could not reach quota, so \(P_1\) is critical.

\(\{\underline{P}_1, P_2, P_3\}\)

Altogether, \(P_1\) is critical 3 times, \(P_2\) is critical 1 time, and \(P_3\) is critical 1 time.

Converting to percents:
\(P_1 = \frac{3}{5} = 60\%\)
\(P_2 = \frac{1}{5} = 20\%\)
\(P_3 = \frac{1}{5} = 20\%\)

Example 5
Consider the voting system \(([16: 7, 6, 3, 3, 2])\). Find the Banzhaf power index.

**Solution**

The winning coalitions are listed below, with the critical players underlined.

\(\{\underline{P}_1, \underline{P}_2, \underline{P}_3\}\)
\(\{\underline{P}_1, \underline{P}_2, \underline{P}_4\}\)
\(\{\underline{P}_1, \underline{P}_2, P_3, P_4\}\)
\(\{\underline{P}_1, \underline{P}_2, \underline{P}_3, P_5\}\)
\(\{\underline{P}_1, \underline{P}_2, \underline{P}_4, P_5\}\)
\(\{\underline{P}_1, \underline{P}_2, P_3, P_4, P_5\}\)
\(\{\underline{P}_1, \underline{P}_2, P_3, P_4, P_5\}\)

\(\{\underline{P}_1, \underline{P}_2, P_3, P_4, P_5\}\)
Counting up times that each player is critical:

\(P_{1}=6\)  
\(P_{2}=6\)  
\(P_{3}=2\)  
\(P_{4}=2\)  
\(P_{5}=0\)

Total of all: 16

Divide each player’s count by 16 to convert to fractions or percents:

\(\frac{P_{1}}{16}=\frac{6}{16}=\frac{3}{8}=37.5\%\)  
\(\frac{P_{2}}{16}=\frac{6}{16}=\frac{3}{8}=37.5\%\)  
\(\frac{P_{3}}{16}=\frac{2}{16}=\frac{1}{8}=12.5\%\)  
\(\frac{P_{4}}{16}=\frac{2}{16}=\frac{1}{8}=12.5\%\)  
\(\frac{P_{5}}{16}=\frac{0}{16}=0=0\%\)

The Banzhaf power index measures a player’s ability to influence the outcome of the vote. Notice that player 5 has a power index of 0, indicating that there is no coalition in which they would be critical power and could influence the outcome. This means player 5 is a dummy, as we noted earlier.

Example 6

Revisiting the Scottish Parliament, with voting system \([65: 47, 46, 17, 16, 2]\), the winning coalitions are listed, with the critical players underlined.

Solution

\(\left\{\underline{P}_{1}, \underline{P}_{2}\right\} \)  
\(\left\{\underline{P}_{1}, \underline{P}_{2}, P_{3}\right\} \)  
\(\left\{\underline{P}_{1}, \underline{P}_{2}, P_{4}\right\} \)  
\(\left\{\underline{P}_{1}, \underline{P}_{2}, P_{5}\right\} \)  
\(\left\{\underline{P}_{1}, \underline{P}_{3}, \underline{P}_{4}\right\} \)  
\(\left\{\underline{P}_{1}, \underline{P}_{3}, \underline{P}_{5}\right\} \)  
\(\left\{\underline{P}_{1}, \underline{P}_{4}, \underline{P}_{5}\right\} \)  
\(\left\{\underline{P}_{2}, \underline{P}_{3}, \underline{P}_{4}\right\} \)  
\(\left\{\underline{P}_{2}, \underline{P}_{3}, \underline{P}_{5}\right\} \)  
\(\left\{P_{1}, P_{2}, P_{3}, P_{4}\right\} \)  
\(\left\{P_{1}, P_{2}, P_{3}, P_{5}\right\} \)  
\(\left\{\underline{P}_{1}, P_{2}, P_{4}, P_{5}\right\} \)
Counting up times that each player is critical:

<table>
<thead>
<tr>
<th>District</th>
<th>Times critical</th>
<th>Power index</th>
</tr>
</thead>
<tbody>
<tr>
<td>P₁ (Scottish National Party)</td>
<td>9</td>
<td>9 / 27 = 33.3%</td>
</tr>
<tr>
<td>P₂ (Labour Party)</td>
<td>7</td>
<td>7 / 27 = 25.9%</td>
</tr>
<tr>
<td>P₃ (Conservative Party)</td>
<td>5</td>
<td>5 / 27 = 18.5%</td>
</tr>
<tr>
<td>P₄ (Liberal Democrats Party)</td>
<td>3</td>
<td>3 / 27 = 11.1%</td>
</tr>
<tr>
<td>P₅ (Scottish Green Party)</td>
<td>3</td>
<td>3 / 27 = 11.1%</td>
</tr>
</tbody>
</table>

Interestingly, even though the Liberal Democrats party has only one less representative than the Conservative Party, and 14 more than the Scottish Green Party, their Banzhaf power index is the same as the Scottish Green Party’s. In parliamentary governments, forming coalitions is an essential part of getting results, and a party’s ability to help a coalition reach quota defines its influence.

Try it Now 3

Find the Banzhaf power index for the weighted voting system \(\textbf{[36: 20, 17, 16, 3]}\).

**Answer**

The voting system tells us that the quota is 36, that Player 1 has 20 votes (or equivalently, has a weight of 20), Player 2 has 17 votes, Player 3 has 16 votes, and Player 4 has 3 votes.

A coalition is any group of one or more players. What we're looking for is winning coalitions - coalitions whose combined votes (weights) add to up to the quota or more. So the coalition \(\{\text{P} 3, \text{P} 4\}\) is not a winning coalition because the combined weight is \(16+3=19\), which is below the quota.

So we look at each possible combination of players and identify the winning ones:

| \{P₁, P₂\} (weight: 37) | \{P₁, P₃\} (weight: 36) | \{P₁, P₂, P₃\} (weight: 53) | \{P₁, P₂, P₄\} (weight: 40) | \{P₁, P₃, P₄\} (weight: 39) | \{P₂, P₃, P₄\} (weight: 36) | \{P₁, P₂, P₃, P₄\} (weight: 56) |

**Example 7**

Banzhaf used this index to argue that the weighted voting system used in the Nassau County Board of Supervisors in New
York was unfair. The county was divided up into 6 districts, each getting voting weight proportional to the population in the district, as shown below. Calculate the power index for each district.

\[
\begin{array}{|l|l|}
\hline \textbf{District} & \textbf{Weight} \\
\hline \text{Hempstead #1} & 31 \\
\hline \text{Hempstead #2} & 31 \\
\hline \text{Oyster Bay} & 28 \\
\hline \text{North Hempstead} & 21 \\
\hline \text{Long Beach} & 2 \\
\hline \text{Glen Cove} & 2 \\
\hline
\end{array}
\]

Solution

Translated into a weighted voting system, assuming a simple majority is needed for a proposal to pass:

\[
\{(58: 31, 31, 28, 21, 2, 2)\}
\]

Listing the winning coalitions and marking critical players:

\[
\begin{align*}
\{\underline{\text{H} 1}, \underline{\text{H} 2}\} & \quad \{\underline{\text{H} 1}, \underline{\text{OB}}, \text{NH}\} & \quad \{\underline{\text{H} 2}, \underline{\text{OB}}, \text{NH}, \text{LB}\} \\
\{\underline{\text{H} 1}, \underline{\text{OB}}\} & \quad \{\underline{\text{H} 1}, \underline{\text{OB}}, \text{LB}\} & \quad \{\underline{\text{H} 2}, \underline{\text{OB}}, \text{NH}, \text{GC}\} \\
\{\underline{\text{H} 2}, \underline{\text{OB}}\} & \quad \{\underline{\text{H} 1}, \underline{\text{OB}}, \text{GC}\} & \quad \{\underline{\text{H} 2}, \underline{\text{OB}}, \text{LB}, \text{GC}\} \\
\{\underline{\text{H} 1}, \underline{\text{H} 2}, \text{NH}\} & \quad \{\underline{\text{H} 1}, \underline{\text{OB}}, \text{NH}, \text{LB}\} & \quad \{\underline{\text{H} 1}, \underline{\text{OB}}, \text{NH}\} \\
\{\underline{\text{H} 1}, \underline{\text{H} 2}, \text{LB}\} & \quad \{\underline{\text{H} 1}, \text{OB}, \text{NH}, \text{GC}\} & \quad \{\text{H} 1, \text{H} 2, \text{OB}\} \\
\{\underline{\text{H} 1}, \underline{\text{H} 2}, \text{GC}\} & \quad \{\underline{\text{H} 1}, \underline{\text{OB}}, \text{LB}, \text{GC}\} & \quad \{\text{H} 1, \text{H} 2, \text{OB}, \text{NH}\} \\
\{\underline{\text{H} 1}, \underline{\text{H} 2}, \text{NH}, \text{LB}\} & \quad \{\underline{\text{H} 1}, \underline{\text{OB}}, \text{NH}, \text{LB}, \text{GC}\} & \quad \{\text{H} 1, \text{H} 2, \text{OB}, \text{NH}, \text{LB}\}
\end{align*}
\]
There are a lot of them! Counting up how many times each player is critical,

\[
\begin{array}{|l|l|l|}
\hline
\textbf{District} & \textbf{Times critical} & \textbf{Power index} \\
\hline
\text{Hempstead #1} & 16 & 16/48=1/3=33\% \\
\hline
\text{Hempstead #2} & 16 & 16/48=1/3=33\% \\
\hline
\text{Oyster Bay} & 16 & 16/48=1/3=33\% \\
\hline
\text{North Hempstead} & 0 & 0/48=0\% \\
\hline
\text{Long Beach} & 0 & 0/48=0\% \\
\hline
\text{Glen Cove} & 0 & 0/48=0\% \\
\hline
\end{array}
\]

It turns out that the three smaller districts are dummies. Any winning coalition requires two of the larger districts.

The weighted voting system that Americans are most familiar with is the Electoral College system used to elect the President. In the Electoral College, states are given a number of votes equal to the number of their congressional representatives (House + Senate). Most states give all their electoral votes to the candidate that wins a majority in their state, turning the Electoral College into a weighted voting system, in which the states are the players. As I’m sure you can imagine, there are billions of possible winning coalitions, so the power index for the Electoral College has to be computed by a computer using approximation techniques.