3.5: Calculating Power- Shapley-Shubik Power Index

The Shapley-Shubik power index was introduced in 1954 by economists Lloyd Shapley and Martin Shubik, and provides a different approach for calculating power.

In situations like political alliances, the order in which players join an alliance could be considered the most important consideration. In particular, if a proposal is introduced, the player that joins the coalition and allows it to reach quota might be considered the most essential. The Shapley-Shubik power index counts how likely a player is to be **pivotal**. What does it mean for a player to be pivotal?

First, we need to change our approach to coalitions. Previously, the coalition \(\{P_{1}, P_{2}\}\) and \(\{P_{2}, P_{1}\}\) would be considered equivalent, since they contain the same players. We now need to consider the **order** in which players join the coalition. For that, we will consider **sequential coalitions** – coalitions that contain all the players in which the order players are listed reflect the order they joined the coalition. For example, the sequential coalition \(\langle P_{2}, P_{1}, P_{3} \rangle\) would mean that \(P_{2}\) joined the coalition first, then \(P_{1}\), and finally \(P_{3}\). The angle brackets \(<>\) are used instead of curly brackets to distinguish sequential coalitions.

**Pivotal Player**

A sequential coalition lists the players in the order in which they joined the coalition.

A **pivotal player** is the player in a sequential coalition that changes a coalition from a losing coalition to a winning one. Notice there can only be one pivotal player in any sequential coalition.

**Example 8**
In the weighted voting system \([8: 6, 4, 3, 2]\), which player is pivotal in the sequential coalition \(<P_{3}, P_{2}, P_{4}, P_{1}>\)?

**Solution**

The sequential coalition shows the order in which players joined the coalition. Consider the running totals as each player joins:

\[
\begin{array}{ll}
P_{3} & \text{Total weight: } 3 & \text{Not winning} \\
P_{3}, P_{2} & \text{Total weight: } 3+4=7 & \text{Not winning} \\
P_{3}, P_{2}, P_{4} & \text{Total weight: } 3+4+2=9 & \text{Winning} \\
P_{3}, P_{2}, P_{4}, P_{1} & \text{Total weight: } 3+4+2+6=15 & \text{Winning}
\end{array}
\]

Since the coalition becomes winning when \(P_4\) joins, \(P_4\) is the pivotal player in this coalition.

**Calculating Shapley-Shubik Power Index**

1. List all sequential coalitions
2. In each sequential coalition, determine the pivotal player
3. Count up how many times each player is pivotal
4. Convert these counts to fractions or decimals by dividing by the total number of sequential coalitions

How many sequential coalitions should we expect to have? If there are \(N\) players in the voting system, then there are \(\binom{N}{1}\) possibilities for the first player in the coalition, \(\binom{N-1}{1}\) possibilities for the second player in the coalition, and so on.

Combining these possibilities, the total number of coalitions would be: \(\binom{N}{1}(N-1)(N-2)(N-3) \cdots (3)(2)(1)\). This calculation is called a **factorial**, and is notated \(\binom{N}{1}\)! The number of sequential coalitions with \(\binom{N}{1}\) players is \(\binom{N}{1}\)

**Example 9**

How many sequential coalitions will there be in a voting system with 7 players?

**Solution**

There will be \(\binom{7}{1}\) sequential coalitions. \(\binom{7}{1}=7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040\)

As you can see, computing the Shapley-Shubik power index by hand would be very difficult for voting systems that are not very small.

**Example 10**

Consider the weighted voting system \([6: 4, 3, 2]\). We will list all the sequential coalitions and identify the pivotal player. We will have \(3! = 6\) sequential coalitions. The coalitions are listed, and the pivotal player is underlined.

\[
\begin{align*}
\text{(array)} & \begin{array}{ll}
P_{3} & \text{Total weight: } 3 \\
P_{3}, P_{2} & \text{Total weight: } 3+4=7 \\
P_{3}, P_{2}, P_{4} & \text{Total weight: } 3+4+2=9 \\
P_{3}, P_{2}, P_{4}, P_{1} & \text{Total weight: } 3+4+2+6=15
\end{array}
\end{align*}
\]
Solution

\(\text{P}_1\) is pivotal 4 times, \(\text{P}_2\) is pivotal 1 time, and \(\text{P}_3\) is pivotal 1 time.

\[
\begin{array}{|l|l|l|}
\hline
\textbf{Player} & \textbf{Times pivotal} & \textbf{Power index} \\
\hline
\text{P}_1 & 4 & 4/6=66.7\% \\
\text{P}_2 & 1 & 1/6=16.7\% \\
\text{P}_3 & 1 & 1/6=16.7\% \\
\hline
\end{array}
\]

For comparison, the Banzhaf power index for the same weighted voting system would be \(\text{P}_1: 60\%, \text{P}_2: 20\%, \text{P}_3: 20\%). While the Banzhaf power index and Shapley-Shubik power index are usually not terribly different, the two different approaches usually produce somewhat different results.

Try it Now 4

Find the Shapley-Shubik power index for the weighted voting system \(\bf{[36: 20, 17, 15]}\).

Answer

Listing all sequential coalitions and identifying the pivotal player:

\[
\begin{array}{llllll}
\text{P}_1, \underline{P}_2, \text{P}_3 & \text{P}_1, \text{P}_3, \underline{P}_2 & \underline{P}_1, \text{P}_2, \text{P}_3 & \text{P}_2, \underline{P}_1, \text{P}_3 & \text{P}_2, \text{P}_3, \underline{P}_1 & \text{P}_3, \underline{P}_1, \text{P}_2 & \text{P}_3, \text{P}_2, \underline{P}_1 & \text{P}_3, \underline{P}_1, \text{P}_2
\end{array}
\]

\(\text{P}_1\) is pivotal 3 times, \(\text{P}_2\) is pivotal 3 times, and \(\text{P}_3\) is pivotal 0 times.

\[
\begin{array}{|l|l|l|}
\hline
\textbf{Player} & \textbf{Times pivotal} & \textbf{Power index} \\
\hline
\text{P}_1 & 3 & 3/6=50\% \\
\text{P}_2 & 3 & 3/6=50\% \\
\text{P}_3 & 0 & 0/6=0\% \\
\hline
\end{array}
\]