3.E: Derivatives (Exercises)

3.1: Defining the Derivative

For the following exercises, use Equation to find the slope of the secant line between the values \(x_1\) and \(x_2\) for each function \(y=f(x)\).

1) \(f(x)=4x+7;\ x_1=2,x_2=5\)

Solution: \(4\)

2) \(f(x)=8x−3;x_1=−1,x_2=3\)

3) \(f(x)=x^2+2x+1;x_1=3,x_2=3.5\)

Solution: \(8.5\)

4) \(f(x)=x^2+2x+2;x_1=0.5,x_2=1.5\)

5) \(f(x)=\frac{3x−1}{4};x_1=1,x_2=3\)

Solution: \(−\frac{3}{4}\)

6) \(f(x)=\frac{x−7}{2x+1};x_1=−2,x_2=0\)

7) \(f(x)=\sqrt{x};x_1=1,x_2=16\)
Solution: \((0.2)\)

8) \((f(x)=\sqrt{x-9}; x_1=10, x_2=13)\)

9) \((f(x)=x^{1/3}+1; x_1=0, x_2=8)\)

Solution: \(0.25\)

10) \((f(x)=6x^{2/3}+2x^{1/3}; x_1=1, x_2=27)\)

For the following functions,

a. use Equation to find the slope of the tangent line \((m_{tan}=f'(a))\), and

b. find the equation of the tangent line to \((f)\) at \((x=a)\).

11) \((f(x)=3-4x, a=2)\)

Solution: \((a. -4) \ (b. y=3-4x)\)

12) \((f(x)=\frac{x}{5}+6, a=-1)\)

13) \((f(x)=x^2+x, a=1)\)

Solution: \((a. 3) \ (b. y=3x-1)\)

14) \((f(x)=1-x-x^2, a=0)\)

15) \((f(x)=\frac{7}{x}, a=3)\)

Solution: \((a. \frac{-7}{9}) \ (b. y=\frac{-7}{9}x+\frac{14}{3})\)

16) \((f(x)=\sqrt{x+8}, a=1)\)

17) \((f(x)=2-3x^2, a=-2)\)

Solution: \((a. 12 b. y=12x+14)\)

18) \((f(x)=\frac{-3}{x-1}, a=4)\)

19) \((f(x)=\frac{2}{x+3}, a=-4)\)

Solution: \((a. -2 b. y=-2x-10)\)

20) \((f(x)=\frac{3}{x^2}, a=3)\)
For the following functions \((y=f(x))\), find \((f'(a))\) using Equation.

21) \((f(x)=5x+4,a=-1)\)

Solution: \((5)\)

22) \((f(x)=-7x+1,a=3)\)

23) \((f(x)=x^2+9x,a=2)\)

Solution: \((13)\)

24) \((f(x)=3x^2-2x+2,a=1)\)

25) \((f(x)=\sqrt{x},a=4)\)

Solution: \((\frac{1}{4})\)

26) \((f(x)=\sqrt{x-2},a=6)\)

27) \((f(x)=\frac{1}{x},a=2)\)

Solution: \((\frac{1}{4})\)

28) \((f(x)=\frac{1}{x-3},a=-1)\)

29) \((f(x)=\frac{1}{x^3},a=1)\)

Solution: \((-1)\)

30) \((f(x)=\frac{1}{\sqrt{x}},a=4)\)

For the following exercises, given the function \((y=f(x))\),

a. find the slope of the secant line \((PQ)\) for each point \((Q(x,f(x)))\) with \((x)\) value given in the table.

b. Use the answers from a. to estimate the value of the slope of the tangent line at \((P)\).

c. Use the answer from b. to find the equation of the tangent line to \((f)\) at point \((P)\).

31) \([T] \ (f(x)=x^2+3x+4), \ (P(1,8))\) (Round to \((6)\) decimal places.)

\[
\begin{array}{llll}
\text{(x)} & \text{(Slope m_{(PQ)})} & \text{(x)} & \text{(Slope m_{(PQ)})} \\
1.1 & (i) & 0.9 & (vii)
\end{array}
\]
Solution:

\(a. \ (i)5.100000, \ (ii)5.010000, \ (iii)5.001000, \ (iv)5.000100, \ (v)5.000010, \ (vi)5.000001, \ (vii)4.900000, \ (viii)4.990000, \ (ix)4.999000, \ (x)4.999900, \ (xi)4.999990, \ (xii)4.999999\)

b. \(m_{\tan}=5\)

c. \(y=5x+3\)

32) [T] \(f(x)=\frac{x+1}{x^2−1}, P(0,−1)\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(\text{Slope } m_{PQ})</th>
<th>(x)</th>
<th>(\text{Slope } m_{PQ})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>(i)</td>
<td>−0.1</td>
<td>(vii)</td>
</tr>
<tr>
<td>0.01</td>
<td>(ii)</td>
<td>−0.01</td>
<td>(viii)</td>
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<tr>
<td>0.001</td>
<td>(iii)</td>
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<td>(ix)</td>
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<tr>
<td>0.0001</td>
<td>(iv)</td>
<td>−0.0001</td>
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<tr>
<td>0.00001</td>
<td>(v)</td>
<td>−0.00001</td>
<td>(xi)</td>
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<tr>
<td>0.000001</td>
<td>(vi)</td>
<td>−0.000001</td>
<td>(xii)</td>
</tr>
</tbody>
</table>

33) [T] \(f(x)=10e^{0.5x}, P(0,10)\) (Round to \(4\) decimal places.)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(\text{Slope } m_{PQ})</th>
</tr>
</thead>
<tbody>
<tr>
<td>−0.1</td>
<td>(i)</td>
</tr>
<tr>
<td>−0.01</td>
<td>(ii)</td>
</tr>
<tr>
<td>−0.001</td>
<td>(iii)</td>
</tr>
<tr>
<td>−0.0001</td>
<td>(iv)</td>
</tr>
</tbody>
</table>
34) \([f(x) = \tan(x)], (P(\pi, 0))\)

\[\begin{array}{cc}
\text{(x)} & \text{(Slope } m_{(PQ)}) \\
3.1 & (i) \\
3.14 & (ii) \\
3.141 & (iii) \\
3.1415 & (iv) \\
3.14159 & (v) \\
3.141592 & (vi)
\end{array}\]

[T] For the following position functions \(y = s(t)\), an object is moving along a straight line, where \(t\) is in seconds and \(s\) is in meters. Find

a. the simplified expression for the average velocity from \(t=2\) to \(t=2+h\);

b. the average velocity between \(t=2\) and \(t=2+h\), where \((i)h=0.1, (ii)h=0.01, (iii)h=0.001\), and \((iv)h=0.0001\);

and

c. use the answer from a. to estimate the instantaneous velocity at \(t=2\) second.

35) \(s(t) = \frac{1}{3}t + 5\)

Solution: \(a. \ (\text{frac} \{1\} \{3\}t+5)\)

Solution: \(a. \ (\text{frac} \{1\} \{3\})\); b. \((i)0.3\) m/s, \((ii)0.3\) m/s, \((iii)0.3\) m/s, \((iv)0.3\) m/s; c. \((0.3=13)\) m/s

36) \(s(t) = t^2 - 2t\)

37) \(s(t) = 2t^3 + 3t\)

Solution: a. \((2(h^2+6h+12))\);
b. \( \text{(i) 25.22 m/s, (ii) 24.12 m/s, (iii) 24.01 m/s, (iv) 24 m/s; c. 24 m/s} \)

38) \( s(t) = \frac{16}{t^2} - \frac{4}{t} \)

39) Use the following graph to evaluate a. \( f'(1) \) and b. \( f'(6) \)

[Graph]

Solution: \( \text{a. 1.25; b. 0.5} \)

40) Use the following graph to evaluate a. \( f'(-3) \) and b. \( f'(1.5) \).

[Graph]

For the following exercises, use the limit definition of derivative to show that the derivative does not exist at \( x = a \) for each of the given functions.
41) \( f(x) = x^{1/3}, \ x=0 \)  
Solution: \( \lim_{x \to 0^-} \frac{x^{1/3}-0}{x-0} = \lim_{x \to 0^-} \frac{1}{x^{2/3}} = \infty \) 

42) \( f(x) = x^{2/3}, \ x=0 \)  

43) \( f(x) = \begin{cases} 1 & x<1 \\ x & x \geq 1 \end{cases}, \ x=1 \)  
Solution: \( \lim_{x \to 1^-} \frac{1-1}{x-1} = 0 \neq 1 = \lim_{x \to 1^+} \frac{x-1}{x-1} \) 

44) \( f(x) = \frac{|x|}{x}, \ x=0 \)  

45) [T] The position in feet of a race car along a straight track after \( t \) seconds is modeled by the function \( s(t) = 8t^2 - \frac{1}{16}t^3. \)  
a. Find the average velocity of the vehicle over the following time intervals to four decimal places:  
i. \([4, 4.1]\)  
ii. \([4, 4.01]\)  
iii. \([4, 4.001]\)  
iv. \([4, 4.0001]\)  
b. Use a. to draw a conclusion about the instantaneous velocity of the vehicle at \( t=4 \) seconds.  
Solution: \( a. \ (i) \) 61.7244 ft/s, \( (ii) \) 61.0725 ft/s, \( (iii) \) 61.0072 ft/s, \( (iv) \) 61.0007 ft/s \)  
b. At \( 4 \) seconds the race car is traveling at a rate/velocity of \( 61 \) ft/s. 

46) [T] The distance in feet that a ball rolls down an incline is modeled by the function \( s(t) = 14t^2, \) where \( t \) is seconds after the ball begins rolling.  
a. Find the average velocity of the ball over the following time intervals:  
i. \([5, 5.1]\)  
ii. \([5, 5.01]\)  
iii. \([5, 5.001]\)  
iv. \([5, 5.0001]\)  
b. Use the answers from a. to draw a conclusion about the instantaneous velocity of the ball at \( t=5 \) seconds.
47) Two vehicles start out traveling side by side along a straight road. Their position functions, shown in the following graph, are given by \(s=f(t)\) and \(s=g(t)\), where \(s\) is measured in feet and \(t\) is measured in seconds.

![Graph showing two position functions, \(f(t)\) and \(g(t)\).]

a. Which vehicle has traveled farther at \(t=2\) seconds?

b. What is the approximate velocity of each vehicle at \(t=3\) seconds?

c. Which vehicle is traveling faster at \(t=4\) seconds?

d. What is true about the positions of the vehicles at \(t=4\) seconds?

Solution:
a. The vehicle represented by \(f(t)\), because it has traveled \(2\) feet, whereas \(g(t)\) has traveled \(1\) foot.

b. The velocity of \(f(t)\) is constant at \(1\) ft/s, while the velocity of \(g(t)\) is approximately \(2\) ft/s.

c. The vehicle represented by \(g(t)\), with a velocity of approximately \(4\) ft/s.

d. Both have traveled \(4\) feet in \(4\) seconds.

48) [T] The total cost \(C(x)\), in hundreds of dollars, to produce \(x\) jars of mayonnaise is given by \(C(x)=0.000003x^3+4x+300\).

a. Calculate the average cost per jar over the following intervals:

i. [100, 100.1]

ii. [100, 100.01]
iii. [100, 100.001]

iv. [100, 100.0001]

b. Use the answers from a. to estimate the average cost to produce 100 jars of mayonnaise.

49) [T] For the function \(f(x)=x^3−2x^2−11x+12\), do the following.

a. Use a graphing calculator to graph \(f\) in an appropriate viewing window.

b. Use the ZOOM feature on the calculator to approximate the two values of \(x=a\) for which \(m_{tan}=f'(a)=0\).

Solution: a.

\[
\begin{align*}
\text{Graph of } f(x) & \text{ in an appropriate viewing window.}
\end{align*}
\]

b. \((a≈-1.361, 2.694)\)

50) [T] For the function \(f(x)=\frac{x}{1+x^2}\), do the following.

a. Use a graphing calculator to graph \(f\) in an appropriate viewing window.

b. Use the ZOOM feature on the calculator to approximate the values of \(x=a\) for which \(m_{tan}=f'(a)=0\).

51) Suppose that \(N(x)\) computes the number of gallons of gas used by a vehicle traveling \(x\) miles. Suppose the vehicle gets 30 mpg.

a. Find a mathematical expression for \(N(x)\).
b. What is $\lambda(N(100))$? Explain the physical meaning.

c. What is $\lambda(N'(100))$? Explain the physical meaning.

Solution: a. $\lambda(N(x)) = \frac{x}{30}$

b. $\sim(3.3)$ gallons. When the vehicle travels $100$ miles, it has used $3.3$ gallons of gas.

c. $\lambda(\frac{1}{30})$. The rate of gas consumption in gallons per mile that the vehicle is achieving after having traveled $100$ miles.

52) \([T]\) For the function $f(x) = x^4 - 5x^2 + 4$, do the following.

a. Use a graphing calculator to graph $f$ in an appropriate viewing window.

b. Use the \(\text{nDeriv}\) function, which numerically finds the derivative, on a graphing calculator to estimate $f'(-2), f'(-0.5), f'(1.7)$, and $f'(2.718)$.

53) \([T]\) For the function $f(x) = \frac{x^2}{x^2 + 1}$, do the following.

a. Use a graphing calculator to graph $f$ in an appropriate viewing window.

b. Use the \(\text{nDeriv}\) function on a graphing calculator to find $f'(-4), f'(-2), f'(2)$, and $f'(4)$.

Solution: a.

b. $(-0.028, -0.16, 0.16, 0.028)$
3.2: The Derivative as a Function

For the following exercises, use the definition of a derivative to find \( f'(x) \).

1) \( f(x) = 6 \)

Solution: \( f'(x) = 0 \)

2) \( f(x) = 2 - 3x \)

Solution: \( f'(x) = -3 \)

3) \( f(x) = \frac{2x}{7} + 1 \)

4) \( f(x) = 4x^2 \)

Solution: \( f'(x) = 8x \)

5) \( f(x) = 5x - x^2 \)

6) \( f(x) = \sqrt{2x} \)

Solution: \( f'(x) = \frac{1}{\sqrt{2x}} \)

7) \( f(x) = \sqrt{x-6} \)

8) \( f(x) = \frac{9}{x} \)

Solution: \( f'(x) = \frac{-9}{x^2} \)

9) \( f(x) = x + \frac{1}{x} \)

10) \( f(x) = \frac{1}{\sqrt{x}} \)

Solution: \( f'(x) = \frac{-1}{2x^{3/2}} \)

For the following exercises, use the graph of \( y = f(x) \) to sketch the graph of its derivative \( f'(x) \).

11)
Solution:
Solution:

For the following exercises, the given limit represents the derivative of a function \( y = f(x) \) at \( x = a \). Find \( f(x) \) and \( f'(a) \).

15) \( \lim_{h \to 0} \frac{(1+h)^{2/3} - 1}{h} \)

16) \( \lim_{h \to 0} \frac{[3(2+h)^2+2]-14}{h} \)
Solution: \( f(x) = 3x^2 + 2, \ a = 2 \)

17) \( \lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h} \)

18) \( \lim_{h \to 0} \frac{(2 + h)^4 - 16}{h} \)

Solution: \( f(x) = x^4, \ a = 2 \)

19) \( \lim_{h \to 0} \frac{[2(3+h)^2 - (3+h)] - 15}{h} \)

20) \( \lim_{h \to 0} \frac{e^h - 1}{h} \)

Solution: \( f(x) = e^x, \ a = 0 \)

For the following functions,

a. sketch the graph and

b. use the definition of a derivative to show that the function is not differentiable at \( x = 1 \).

21) \( f(x) = \begin{cases} 2\sqrt{x} & 0 \leq x \leq 1 \\ 3x - 1 & x > 1 \end{cases} \)

22) \( f(x) = \begin{cases} 3 & x < 1 \\ 3x & x \geq 1 \end{cases} \)

Solution:

a.
b. \(\lim_{h\to 1^-} \frac{3-3}{h} \neq \lim_{h\to 1^+} \frac{3h}{h}\)

23) \(f(x) = \begin{cases} -x^2 + 2 & x \leq 1 \\&\ x > 1 \end{cases}\)

24) \(f(x) = \begin{cases} 2x, & x \leq 1 \\&\ \frac{2}{x} & x > 1 \end{cases}\)

a. 

b. \(\lim_{h\to 1^-} \frac{2h}{h} \neq \lim_{h\to 1^+} \frac{\frac{2}{x+h} - 2x}{h}\).

For the following graphs,

a. determine for which values of \(x=a\) the \(\lim_{x\to a} f(x)\) exists but \(f\) is not continuous at \(x=a\), and

b. determine for which values of \(x=a\) the function is continuous but not differentiable at \(x=a\).

25)
26) Solution: \(a. x=1, b. x=2\)

27) Use the graph to evaluate \(a. f'(-0.5), b. f'(0), c. f'(1), d. f'(2), e. f'(3)\) if it exists.
For the following functions, use \(f''(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\) to find \(f''(x)\).

28) \(f(x) = 2 - 3x\)

Solution: \(0\)

29) \(f(x) = 4x^2\)

30) \(f(x) = x + \frac{1}{x}\)

Solution: \(\frac{2}{x^3}\)

For the following exercises, use a calculator to graph \(f(x)\). Determine the function \(f'(x)\), then use a calculator to graph \(f'(x)\).

31) \(f(x) = -\frac{5}{x}\)

Solution: \(f'(x) = \frac{5}{x^2}\)

32) \(f(x) = 3x^2 + 2x + 4\)

Solution: \(f'(x) = 6x + 2\)
33) \( f(x) = \sqrt{x} + 3x \)

34) \( f(x) = \frac{1}{\sqrt{2x}} \)

Solution: \( f'(x) = -\frac{1}{(2x)^{3/2}} \)

35) \( f(x) = 1 + x + \frac{1}{x} \)

36) \( f(x) = x^3 + 1 \)

Solution: \( f'(x) = 3x^2 \)
For the following exercises, describe what the two expressions represent in terms of each of the given situations. Be sure to include units.

a. \( \frac{f(x+h)−f(x)}{h} \)

b. \( f′(x)=\lim_{h→0}\frac{f(x+h)−f(x)}{h} \)

37) \( P(x) \) denotes the population of a city at time \( x \) in years.

38) \( C(x) \) denotes the total amount of money (in thousands of dollars) spent on concessions by \( x \) customers at an amusement park.

Solution:

a. Average rate at which customers spent on concessions in thousands per customer.

b. Rate (in thousands per customer) at which \( x \) customers spent money on concessions in thousands per customer.

39) \( R(x) \) denotes the total cost (in thousands of dollars) of manufacturing \( x \) clock radios

40) \( g(x) \) denotes the grade (in percentage points) received on a test, given \( x \) hours of studying.

a. Average grade received on the test with an average study time between two values.

b. Rate (in percentage points per hour) at which the grade on the test increased or decreased for a given average study time of \( x \) hours.
41) \(B(x)\) denotes the cost (in dollars) of a sociology textbook at university bookstores in the United States in \(x\) years since \(1990\).

42) \(p(x)\) denotes atmospheric pressure at an altitude of \(x\) feet.

Solution:

a. Average change of atmospheric pressure between two different altitudes.

b. Rate (torr per foot) at which atmospheric pressure is increasing or decreasing at \(x\) feet.

43) Sketch the graph of a function \(y=f(x)\) with all of the following properties:

a. \(f'(x)>0\) for \((-2\leq x<1)\)

b. \(f'(2)=0\)

c. \(f'(x)>0\) for \(x>2\)

d. \(f(2)=2\) and \(f(0)=1\)

e. \(\lim_{x\to-\infty}f(x)=0\) and \(\lim_{x\to\infty}f(x)=\infty\)

f. \(f'(1)\) does not exist.

44) Suppose temperature \(T\) in degrees Fahrenheit at a height \(x\) in feet above the ground is given by \(y=T(x)\).

a. Give a physical interpretation, with units, of \(T'(x)\).

b. If we know that \(T'(1000)=-0.1\), explain the physical meaning.

Solution:

a. The rate (in degrees per foot) at which temperature is increasing or decreasing for a given height \(x\).

b. The rate of change of temperature as altitude changes at \(1000\) feet is \(-0.1\) degrees per foot.

45) Suppose the total profit of a company is \(y=P(x)\) thousand dollars when \(x\) units of an item are sold.

a. What does \( \frac{P(b)-P(a)}{b-a} \) for \(0<a<b\) measure, and what are the units?

b. What does \( P'(x) \) measure, and what are the units?

c. Suppose that \( P'(30)=5 \), what is the approximate change in profit if the number of items sold increases from \(30\) to \(31\)?
46) The graph in the following figure models the number of people \( N(t) \) who have come down with the flu \( t \) weeks after its initial outbreak in a town with a population of 50,000 citizens.

a. Describe what \( N'(t) \) represents and how it behaves as \( t \) increases.

b. What does the derivative tell us about how this town is affected by the flu outbreak?

Solution: a. The rate at which the number of people who have come down with the flu is changing \( t \) weeks after the initial outbreak. b. The rate is increasing sharply up to the third week, at which point it slows down and then becomes constant.

For the following exercises, use the following table, which shows the height \( h \) of the Saturn \( V \) rocket for the Apollo \( 11 \) mission \( t \) seconds after launch.

<table>
<thead>
<tr>
<th>Time(seconds)</th>
<th>Height(meters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
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<td>4</td>
<td>25</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
</tr>
</tbody>
</table>

47) What is the physical meaning of \( h'(t) \)? What are the units?

48) [T] Construct a table of values for \( h'(t) \) and graph both \( h(t) \) and \( h'(t) \) on the same graph. (Hint: for interior points,
estimate both the left limit and right limit and average them.)

Solution:

<table>
<thead>
<tr>
<th>Time (seconds)</th>
<th>( h'(t) ) (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>5.5</td>
</tr>
<tr>
<td>3</td>
<td>10.5</td>
</tr>
<tr>
<td>4</td>
<td>9.5</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
</tr>
</tbody>
</table>

49) [T] The best linear fit to the data is given by \( H(t) = 7.229t - 4.905 \), where \( H(t) \) is the height of the rocket (in meters) and \( t \) is the time elapsed since takeoff. From this equation, determine \( H'(t) \). Graph \( H(t) \) with the given data and, on a separate coordinate plane, graph \( H'(t) \).

50) [T] The best quadratic fit to the data is given by \( G(t) = 1.429t^2 + 0.0857t - 0.1429 \), where \( G(t) \) is the height of the rocket (in meters) and \( t \) is the time elapsed since takeoff. From this equation, determine \( G'(t) \). Graph \( G(t) \) with the given data and, on a separate coordinate plane, graph \( G'(t) \).

Solution: \( G'(t) = 2.858t + 0.0857 \)
51) [T] The best cubic fit to the data is given by \(F(t)=0.2037t^3+2.956t^2−2.705t+0.4683\), where \(F\) is the height of the rocket (in m) and \(t\) is the time elapsed since take off. From this equation, determine \(F′(t)\). Graph \(F(t)\) with the given data and, on a separate coordinate plane, graph \(F′(t)\). Does the linear, quadratic, or cubic function fit the data best?

52) Using the best linear, quadratic, and cubic fits to the data, determine what \(H''(t), G''(t)\) and \(F''(t)\) are. What are the physical meanings of \(H''(t), G''(t)\) and \(F''(t)\), and what are their units?

Solution: \(H''(t)=0, G''(t)=2.858\) and \(f''(t)=1.222t+5.912\) represent the acceleration of the rocket, with units of meters per second squared \((m/s^2)\).

3.3: Differentiation Rules

For the following exercises, find \(f′(x)\) for each function.

1) \(f(x)=x^7+10\)

2) \(f(x)=5x^3−x+1\)

Solution: \(f′(x)=15x^2−1\)

3) \(f(x)=4x^2−7x\)

4) \(f(x)=8x^4+9x^2−1\)

Solution: \(f′(x)=32x^3+18x\)

5) \(f(x)=x^4+2x\)

6) \(f(x)=3x(18x^4+\frac{13}{x+1})\)

Solution: \(f′(x)=270x^4+\frac{13}{(x+1)^2}\)
7) \(f(x) = (x+2)(2x^2−3)\)

8) \(f(x) = x^2(\frac{2}{x^2}+\frac{5}{x^3})\)

Solution: \(f'(x) = -\frac{5}{x^2}\)

9) \(f(x) = x^3+2x^2−4\)

10) \(f(x) = 4x^3−2x+1\)

Solution: \(f'(x) = 4x^2+2x^2−2x\)

11) \(f(x) = x^2+4\)

12) \(f(x) = x+9\)

Solution: \(f'(x) = -x^2−18x+64\)

For the following exercises, find the equation of the tangent line \(T(x)\) to the graph of the given function at the indicated point. Use a graphing calculator to graph the function and the tangent line.

13) \(y = 3x^2+4x+1\) at \((0,1)\)

14) \(y = 2\sqrt{x}+1\) at \((4,5)\)

Solution: \(T(x) = \frac{1}{2}x+3\)

15) \(y = \frac{2x}{x−1}\) at \((-1,1)\)
16) \( y = \frac{2}{x} - \frac{3}{x^2} \) at \((1, -1)\)

Solution: \( T(x) = 4x - 5 \)

For the following exercises, assume that \( f(x) \) and \( g(x) \) are both differentiable functions for all \( x \). Find the derivative of each of the functions \( h(x) \).

17) \( h(x) = 4f(x) + \frac{g(x)}{7} \)

Solution: \( h'(x) = 4f'(x) + \frac{g'(x)}{7} \)

18) \( h(x) = x^3f(x) \)

Solution: \( h'(x) = 3x^2f(x) + x^3f'(x) \)

19) \( h(x) = \frac{f(x)g(x)}{2} \)

20) \( h(x) = \frac{3f(x)}{g(x)+2} \)

Solution: \( h'(x) = \frac{3f'(x)(g(x)+2) - 3f(x)g'(x)}{(g(x)+2)^2} \)

For the following exercises, assume that \( f(x) \) and \( g(x) \) are both differentiable functions with values as given in the following table. Use the following table to calculate the following derivatives.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( f'(x) )</th>
<th>( g'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>2</td>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>3</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
<td>-4</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>6</td>
<td>-3</td>
<td>9</td>
</tr>
</tbody>
</table>

21) Find \( h'(1) \) if \( h(x) = x f(x) + 4g(x) \).

22) Find \( h'(2) \) if \( h(x) = \frac{f(x)}{g(x)} \).
23) Find $h'(3)$ if $h(x) = 2x + f(x)g(x)$.

24) Find $h'(4)$ if $h(x) = \frac{1}{x} + \frac{g(x)}{f(x)}$.

Solution: Undefined

For the following exercises, use the following figure to find the indicated derivatives, if they exist.

25) Let $h(x) = f(x) + g(x)$. Find
   a) $h'(1)$,
   b) $h'(3)$, and
   c) $h'(4)$.

26) Let $h(x) = f(x)g(x)$. Find
   a) $h'(1)$,
   b) $h'(3)$, and
   c) $h'(4)$.

Solution: a. $2$, b. does not exist, c. $2.5$

27) Let $h(x) = \frac{f(x)}{g(x)}$. Find

Solution: $\frac{16}{9}$
a) \( h'(1), \)

b) \( h'(3), \) and

c) \( h'(4). \)

For the following exercises,

a) evaluate \( f'(a), \) and

b) graph the function \( f(x) \) and the tangent line at \( x=a. \)

28) \([T]\) \( f(x)=2x^3+3x-x^2, a=2 \)

Solution: a. 23, b. \( y=23x-28 \)

29) \([T]\) \( f(x)=\frac{1}{x}-x^2, a=1 \)

30) \([T]\) \( f(x)=x^2-x^{12}+3x+2, a=0 \)

Solution: a. 3, b. \( y=3x+2 \)
31) \( f(x) = \frac{1}{x} - x^{2/3}, a = -1 \)

32) Find the equation of the tangent line to the graph of \( f(x) = 2x^3 + 4x^2 - 5x - 3 \) at \( x = -1 \).

Solution: \( y = -7x - 3 \)

33) Find the equation of the tangent line to the graph of \( f(x) = x^2 + \frac{4}{x} - 10 \) at \( x = 8 \).

34) Find the equation of the tangent line to the graph of \( f(x) = (3x - x^2)(3 - x - x^2) \) at \( x = 1 \).

Solution: \( y = -5x + 7 \)

35) Find the point on the graph of \( f(x) = x^3 \) such that the tangent line at that point has an x intercept of 6.

36) Find the equation of the line passing through the point \( P(3,3) \) and tangent to the graph of \( f(x) = 6 / (x - 1) \).

Solution: \( y = -\frac{3}{2}x + \frac{15}{2} \)

37) Determine all points on the graph of \( f(x) = x^3 + x^2 - 2x - 1 \) for which the slope of the tangent line is
38) Find a quadratic polynomial such that \(f(1)=5, f'(1)=3\) and \(f''(1)=-6\).

Solution: \(y=-3x^2+9x-1\)

39) A car driving along a freeway with traffic has traveled \(s(t)=t^3-6t^2+9t\) meters in \(t\) seconds.

   a. Determine the time in seconds when the velocity of the car is 0.

   b. Determine the acceleration of the car when the velocity is 0.

[T] A herring swimming along a straight line has traveled \(s(t)=\frac{t^2}{t^2+2}\) feet in \(t\) seconds.

40) Determine the velocity of the herring when it has traveled 3 seconds.

Solution: \(\frac{12}{121}\) or 0.0992 ft/s

41) The population in millions of arctic flounder in the Atlantic Ocean is modeled by the function \(P(t)=\frac{8t+3}{0.2t^2+1}\), where \(t\) is measured in years.

   a. Determine the initial flounder population.

   b. Determine \(P'(10)\) and briefly interpret the result.

42) [T] The concentration of antibiotic in the bloodstream \(t\) hours after being injected is given by the function \(C(t)=\frac{2t^2+t}{t^3+50}\), where \(C\) is measured in milligrams per liter of blood.

   a. Find the rate of change of \(C(t)\).

   b. Determine the rate of change for \(t=8,12,24\), and \(36\).

   c. Briefly describe what seems to be occurring as the number of hours increases.

Solution: \(a. \frac{-2t^4-2t^3+200t+50}{(t^3+50)^2}\) \(b. -0.02395\) mg/L-hr, \(-0.01344\) mg/L-hr, \(-0.003566\) mg/L-hr, \(-0.001579\) mg/L-hr c. The rate at which the concentration of drug in the bloodstream decreases is slowing to 0 as time increases.

43) A book publisher has a cost function given by \(C(x)=\frac{x^3+2x+3}{x^2}\), where \(x\) is the number of copies of a book in thousands and \(C\) is the cost, per book, measured in dollars. Evaluate \(C'(2)\) and explain its meaning.

44) [T] According to Newton’s law of universal gravitation, the force \(F\) between two bodies of constant mass \(m_1\) and
\(m_2\) is given by the formula \(F=\frac{GM_1m_2}{d^2}\), where \(G\) is the gravitational constant and \(d\) is the distance between the bodies.

a. Suppose that \(G, m_1,\) and \(m_2\) are constants. Find the rate of change of force \(F\) with respect to distance \(d\).

b. Find the rate of change of force \(F\) with gravitational constant \(G=6.67\times10^{-11} \text{ Nm}^2/\text{kg}^2\), on two bodies 10 meters apart, each with a mass of 1000 kilograms.

Solution: \((a. F'(d)=\frac{-2GM_1m_2}{d^3}) \quad (b. -1.33\times10^{-7} \text{ N/m})\)

### 3.4: Derivatives as Rates of Change

For the following exercises, the given functions represent the position of a particle traveling along a horizontal line.

a. Find the velocity and acceleration functions.

b. Determine the time intervals when the object is slowing down or speeding up.

1) \(s(t)=2t^3-3t^2-12t+8\)

2) \(s(t)=2t^3-15t^2+36t-10\)

Solution: a. \((v(t)=6t^2-30t+36, a(t)=12t-30)\); b. speeds up \((2,2.5)?(3,\infty)\), slows down \((0,2)?(2.5,3)\)

3) \(s(t)=\frac{t}{1+t^2}\)

4) A rocket is fired vertically upward from the ground. The distance \(s\) in feet that the rocket travels from the ground after \(t\) seconds is given by \(s(t)=-16t^2+560t\).

a. Find the velocity of the rocket 3 seconds after being fired.

b. Find the acceleration of the rocket 3 seconds after being fired.

Solution: \((a. 464 \text{ ft/s}^2) \quad (b. -32 \text{ ft/s}^2)\)

5) A ball is thrown downward with a speed of 8 ft/s from the top of a 64-foot-tall building. After \(t\) seconds, its height above the ground is given by \(s(t)=-16t^2-8t+64\).

a. Determine how long it takes for the ball to hit the ground.

b. Determine the velocity of the ball when it hits the ground.

6) The position function \(s(t)=t^2-2t-4\) represents the position of the back of a car backing out of a driveway and then
driving in a straight line, where \( s \) is in feet and \( t \) is in seconds. In this case, \( s(t)=0 \) represents the time at which the back of the car is at the garage door, so \( s(0)=-4 \) is the starting position of the car, 4 feet inside the garage.

a. Determine the velocity of the car when \( s(t)=0 \).

b. Determine the velocity of the car when \( s(t)=14 \).

Solution: (a. 5 ft/s) (b. 9 ft/s)

7) The position of a hummingbird flying along a straight line in \( t \) seconds is given by \( s(t)=3t^3-7t \) meters.

a. Determine the velocity of the bird at \( t=1 \) sec.

b. Determine the acceleration of the bird at \( t=1 \) sec.

c. Determine the acceleration of the bird when the velocity equals 0.

8) A potato is launched vertically upward with an initial velocity of 100 ft/s from a potato gun at the top of an 85-foot-tall building. The distance in feet that the potato travels from the ground after \( t \) seconds is given by \( s(t)=-16t^2+100t+85 \).

a. Find the velocity of the potato after \( t=0.5s \) and \( t=5.75s \).

b. Find the speed of the potato at 0.5 s and 5.75 s.

c. Determine when the potato reaches its maximum height.

d. Find the acceleration of the potato at 0.5 s and 1.5 s.

e. Determine how long the potato is in the air.

f. Determine the velocity of the potato upon hitting the ground.

Solution: a. 84 ft/s, -84 ft/s b. 84 ft/s c. \( \frac{25}{8}s \) d. \( (-32ft/s^2) \) in both cases e. \( \frac{1}{8}(25+\sqrt{965})s \) f. \( -4\sqrt{965}ft/s \)

9) The position function \( s(t)=t^3-8t \) gives the position in miles of a freight train where east is the positive direction and \( t \) is measured in hours.

a. Determine the direction the train is traveling when \( s(t)=0 \).

b. Determine the direction the train is traveling when \( a(t)=0 \).

c. Determine the time intervals when the train is slowing down or speeding up.

10) The following graph shows the position \( y=s(t) \) of an object moving along a straight line.
a. Use the graph of the position function to determine the time intervals when the velocity is positive, negative, or zero.

b. Sketch the graph of the velocity function.

c. Use the graph of the velocity function to determine the time intervals when the acceleration is positive, negative, or zero.

d. Determine the time intervals when the object is speeding up or slowing down.

Solution: a. Velocity is positive on \((0,1.5)\) and \((6,7)\), negative on \((1.5,2)\) and \((5,6)\), and zero on \((2,5)\).

b. 
c. Acceleration is positive on \((5,7)\), negative on \((0,2)\), and zero on \((2,5)\). d. The object is speeding up on \((6,7)\) and slowing down on \((0,1.5)\).

11) The cost function, in dollars, of a company that manufactures food processors is given by \(C(x)=200+\frac{7}{x}+\frac{x}{27}\), where \(x\) is the number of food processors manufactured.

a. Find the marginal cost function.

b. Find the marginal cost of manufacturing 12 food processors.

c. Find the actual cost of manufacturing the thirteenth food processor.

12) The price \(p\) (in dollars) and the demand \(x\) for a certain digital clock radio is given by the price–demand function \(p=10−0.001x\).

a. Find the revenue function \(R(x)\)

b. Find the marginal revenue function.

c. Find the marginal revenue at \(x=2000\) and \(5000\).

Solution: a. \(R(x)=10x−0.001x^2\) b. \(R'(x)=10−0.002x\) c. $6 per item, $0 per item

13) [T] A profit is earned when revenue exceeds cost. Suppose the profit function for a skateboard manufacturer is given by \(P(x)=30x−0.3x^2−250\), where \(x\) is the number of skateboards sold.

a. Find the exact profit from the sale of the thirtieth skateboard.

b. Find the marginal profit function and use it to estimate the profit from the sale of the thirtieth skateboard.
14) [T] In general, the profit function is the difference between the revenue and cost functions: \((P(x)=R(x)−C(x))\).

Suppose the price-demand and cost functions for the production of cordless drills is given respectively by \((p=143−0.03x)\) and \((C(x)=75,000+65x)\), where \((x)\) is the number of cordless drills that are sold at a price of \((p)\) dollars per drill and \((C(x))\) is the cost of producing \((x)\) cordless drills.

a. Find the marginal cost function.

b. Find the revenue and marginal revenue functions.

c. Find \((R'(1000))\) and \((R'(4000))\). Interpret the results.

d. Find the profit and marginal profit functions.

e. Find \((P'(1000))\) and \((P'(4000))\). Interpret the results.

Solution: a. \((C'(x)=65)\) b. \((R(x)=143x−0.03x^2)\), \((R'(x)=143−0.06x)\) c. \((83,−97)\). At a production level of 1000 cordless drills, revenue is increasing at a rate of $83 per drill; at a production level of 4000 cordless drills, revenue is decreasing at a rate of $97 per drill. d. \((P(x)=−0.03x^2+78x−75000)\), \((P'(x)=−0.06x+78)\) c. \((18,−162)\). At a production level of 1000 cordless drills, profit is increasing at a rate of $18 per drill; at a production level of 4000 cordless drills, profit is decreasing at a rate of $162 per drill.

15) A small town in Ohio commissioned an actuarial firm to conduct a study that modeled the rate of change of the town’s population. The study found that the town’s population (measured in thousands of people) can be modeled by the function \((P(t)=−\frac{1}{3}t^3+64t+3000)\), where \((t)\) is measured in years.

a. Find the rate of change function \((P'(t))\) of the population function.

b. Find \((P'(1),P'(2),P'(3),\text{ and }P'(4))\). Interpret what the results mean for the town.

c. Find \((P''(1),P''(2),P''(3),\text{ and }P''(4))\). Interpret what the results mean for the town’s population.

16) [T] A culture of bacteria grows in number according to the function \((N(t)=3000(1+\frac{4t}{t^2+100}))\), where \((t)\) is measured in hours.

a. Find the rate of change of the number of bacteria.

b. Find \((N'(0),N'(10),N'(20)),\text{ and }N'(30))\).

c. Interpret the results in (b).

d. Find \((N''(0),N''(10),N''(20)),\text{ and }N''(30))\). Interpret what the answers imply about the bacteria population growth.

Solution: a. \((N'(t)=3000(\frac{−4t^2+400}{(t^2+100)}^2))\) b. \((120,0,−14.4,−9.6)\) c. The bacteria population increases from time 0 to 10 hours; afterwards, the bacteria population decreases. d. \((0,−6,0.384,0.432)\). The rate at which the bacteria is
increasing is decreasing during the first 10 hours. Afterwards, the bacteria population is decreasing at a decreasing rate.

17) The centripetal force of an object of mass m is given by \( F(r) = \frac{mv^2}{r} \), where \( v \) is the speed of rotation and \( r \) is the distance from the center of rotation.

   a. Find the rate of change of centripetal force with respect to the distance from the center of rotation.

   b. Find the rate of change of centripetal force of an object with mass 1000 kilograms, velocity of 13.89 m/s, and a distance from the center of rotation of 200 meters.

The following questions concern the population (in millions) of London by decade in the 19th century, which is listed in the following table.

<table>
<thead>
<tr>
<th>Year Since 1800</th>
<th>Population (millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8975</td>
</tr>
<tr>
<td>11</td>
<td>1.040</td>
</tr>
<tr>
<td>21</td>
<td>1.264</td>
</tr>
<tr>
<td>31</td>
<td>1.516</td>
</tr>
<tr>
<td>41</td>
<td>1.661</td>
</tr>
<tr>
<td>51</td>
<td>2.000</td>
</tr>
<tr>
<td>61</td>
<td>2.634</td>
</tr>
<tr>
<td>71</td>
<td>3.272</td>
</tr>
<tr>
<td>81</td>
<td>3.911</td>
</tr>
<tr>
<td>91</td>
<td>4.422</td>
</tr>
</tbody>
</table>

Population of London


18) [T]

a. Using a calculator or a computer program, find the best-fit linear function to measure the population.

b. Find the derivative of the equation in a. and explain its physical meaning.

c. Find the second derivative of the equation and explain its physical meaning.

Solution: a. \( P(t) = 0.03983 + 0.4280t \) b. \( P'(t) = 0.03983 \). The population is increasing. c. \( P''(t) = 0 \). The rate at which the population is increasing is constant.
19) [T]

a. Using a calculator or a computer program, find the best-fit quadratic curve through the data.

b. Find the derivative of the equation and explain its physical meaning.

c. Find the second derivative of the equation and explain its physical meaning.

For the following exercises, consider an astronaut on a large planet in another galaxy. To learn more about the composition of this planet, the astronaut drops an electronic sensor into a deep trench. The sensor transmits its vertical position every second in relation to the astronaut’s position. The summary of the falling sensor data is displayed in the following table.

<table>
<thead>
<tr>
<th>Time after dropping (s)</th>
<th>Position (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
</tr>
<tr>
<td>3</td>
<td>-5</td>
</tr>
<tr>
<td>4</td>
<td>-7</td>
</tr>
<tr>
<td>5</td>
<td>-14</td>
</tr>
</tbody>
</table>

20) [T]

a. Using a calculator or computer program, find the best-fit quadratic curve to the data.

b. Find the derivative of the position function and explain its physical meaning.

c. Find the second derivative of the position function and explain its physical meaning.

Solution: a. \( p(t) = -0.6071t^2 + 0.4357t - 0.3571 \) b. \( p'(t) = -1.214t + 0.4357 \). This is the velocity of the sensor. c. \( p''(t) = -1.214 \). This is the acceleration of the sensor; it is a constant acceleration downward.

21) [T]

a. Using a calculator or computer program, find the best-fit cubic curve to the data.

b. Find the derivative of the position function and explain its physical meaning.

c. Find the second derivative of the position function and explain its physical meaning.

d. Using the result from c. explain why a cubic function is not a good choice for this problem.
The following problems deal with the Holling type I, II, and III equations. These equations describe the ecological event of growth of a predator population given the amount of prey available for consumption.

22) [T] The **Holling type I equation** is described by \( f(x) = ax \), where \( x \) is the amount of prey available and \( a > 0 \) is the rate at which the predator meets the prey for consumption.

a. Graph the Holling type I equation, given \( a = 0.5 \).

b. Determine the first derivative of the Holling type I equation and explain physically what the derivative implies.

c. Determine the second derivative of the Holling type I equation and explain physically what the derivative implies.

d. Using the interpretations from b. and c. explain why the Holling type I equation may not be realistic.

Solution:

a.

b. \( f'(x) = a \). The more increase in prey, the more growth for predators. c. \( f''(x) = 0 \). As the amount of prey increases, the rate at which the predator population growth increases is constant. d. This equation assumes that if there is more prey, the predator is able to increase consumption linearly. This assumption is unphysical because we would expect there to be some saturation point at which there is too much prey for the predator to consume adequately.

23) [T] The Holling type II equation is described by \( f(x) = \frac{ax}{n+x} \), where \( x \) is the amount of prey available and \( a > 0 \) is the maximum consumption rate of the predator.

a. Graph the Holling type II equation given \( a = 0.5 \) and \( n = 5 \). What are the differences between the Holling type I and II equations?

b. Take the first derivative of the Holling type II equation and interpret the physical meaning of the derivative.
c. Show that \( f(n) = \frac{1}{2}a \) and interpret the meaning of the parameter n.

d. Find and interpret the meaning of the second derivative. What makes the Holling type II function more realistic than the Holling type I function?

24) [T] The Holling type III equation is described by \( f(x) = \frac{ax^2}{n^2+x^2} \), where x is the amount of prey available and \( a > 0 \) is the maximum consumption rate of the predator.

a. Graph the Holling type III equation given \( a=0.5 \) and \( n=5 \). What are the differences between the Holling type II and III equations?

b. Take the first derivative of the Holling type III equation and interpret the physical meaning of the derivative.

c. Find and interpret the meaning of the second derivative (it may help to graph the second derivative).

d. What additional ecological phenomena does the Holling type III function describe compared with the Holling type II function?

Solution:

a.

![Graph of f(x)](image)

\[ f'(x) = \frac{2axn^2}{(n^2+x^2)^2} \] When the amount of prey increases, the predator growth increases.

\[ f''(x) = \frac{2an^2(n^2-3x^2)}{(n^2+x^2)^3} \] When the amount of prey is extremely small, the rate at which predator growth is increasing begins to decrease. At lower levels of prey, the prey is more easily able to avoid detection by the predator, so fewer prey individuals are consumed, resulting in less predator growth.

25) [T] The populations of the snowshoe hare (in thousands) and the lynx (in hundreds) collected over 7 years from 1937 to 1943 are shown in the following table. The snowshoe hare is the primary prey of the lynx.

<table>
<thead>
<tr>
<th>Population of snowshoe hare (thousands)</th>
<th>Population of lynx (hundreds)</th>
</tr>
</thead>
</table>

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Snowshoe Hare and Lynx Populations

Source: http://www.biotopics.co.uk/newgcse/predatorprey.html

a. Graph the data points and determine which Holling-type function fits the data best.

b. Using the meanings of the parameters \(a\) and \(n\), determine values for those parameters by examining a graph of the data. Recall that \(n\) measures what prey value results in the half-maximum of the predator value.

c. Plot the resulting Holling-type I, II, and III functions on top of the data. Was the result from part a. correct?

3.5: Derivatives of Trigonometric Functions

For the following exercises, find \(\frac{dy}{dx}\) for the given functions.

1) \(y=x^2-\sec x+1\)

Solution: \(\frac{dy}{dx}=2x-\sec x\tan x\)

2) \(y=3\csc x+\frac{5}{x}\)

3) \(y=x^2\cot x\)

Solution: \(\frac{dy}{dx}=2xcotx-x^2csc^2x\)

4) \(y=x-x^3\sin x\)

5) \(y=\frac{\sec x}{x}\)

Solution: \(\frac{dy}{dx}=(x\sec x\tan x-\sec x)/x^2\)

6) \(y=\tan x\cot x\)

7) \(y=(x+\cos x)(1-\sin x)\)

Solution: \(\frac{dy}{dx}=(1-\sin x)(1-\sin x) - \cos x(x+\cos x)\)

8) \(y=\frac{\tan x}{1-\sec x}\)
9) \(y=\frac{1-cotx}{1+cotx}\) 

Solution: \(\frac{dy}{dx}=\frac{2csc^2x}{(1+cotx)^2}\)

10) \(y=cosx(1+cscx)\)

For the following exercises, find the equation of the tangent line to each of the given functions at the indicated values of \(x\). Then use a calculator to graph both the function and the tangent line to ensure the equation for the tangent line is correct.

11) \(f(x)=-sinx, x=0\)

Solution: \(y=-x\)

12) \(f(x)=cscx, x=\frac{\pi}{2}\)

13) \(f(x)=1+cosx, x=\frac{3\pi}{2}\)

Solution: \(y=x+\frac{2-3\pi}{2}\)
14) \( f(x) = \sec x, x = \frac{\pi}{4} \)

15) \( f(x) = x^2 - \tan xx = 0 \)

Solution: \( y = -x \)

16) \( f(x) = 5 \cot x, x = \frac{\pi}{4} \)

For the following exercises, find \( \frac{d^2y}{dx^2} \) for the given functions.

17) \( y = x \sin x - \cos x \)

Solution: \( 3 \cos x - x \sin x \)
18) \(y = \sin x \cos x\)

19) \(y = x - \frac{1}{2} \sin x\)

Solution: \(\frac{1}{2} \sin x\)

20) \(y = \frac{1}{x} + \tan x\)

21) \(y = 2 \csc x\)

Solution: \((\csc x)(3 \csc^2 x - 1 + \cot^2 x)\)

22) \(y = \sec^2 x\)

23) Find all \(x\) values on the graph of \(f(x) = -3 \sin x \cos x\) where the tangent line is horizontal.

Solution: \(\frac{(2n+1)\pi}{4}\), where \(n\) is an integer

24) Find all \(x\) values on the graph of \(f(x) = x - 2 \cos x\) for \(0 < x < 2\pi\) where the tangent line has slope 2.

25) Let \(f(x) = \cot x\). Determine the points on the graph of \(f\) for \(0 < x < 2\pi\) where the tangent line(s) is (are) parallel to the line \(y = -2x\).

Solution: \((\frac{\pi}{4}, 1), (\frac{3\pi}{4}, -1)\)

26) [T] A mass on a spring bounces up and down in simple harmonic motion, modeled by the function \(s(t) = -6 \cos t\) where \(s\) is measured in inches and \(t\) is measured in seconds. Find the rate at which the spring is oscillating at \(t = 5\) s.

27) Let the position of a swinging pendulum in simple harmonic motion be given by \(s(t) = a \cos t + b \sin t\). Find the constants \(a\) and \(b\) such that when the velocity is 3 cm/s, \(s(0)\) and \(s(t=0)\).

Solution: \(a = 0, b = 3\)

28) After a diver jumps off a diving board, the edge of the board oscillates with position given by \(s(t) = -5 \cos t\) cm at \(t\) seconds after the jump.

   a. Sketch one period of the position function for \(t \geq 0\).

   b. Find the velocity function.

   c. Sketch one period of the velocity function for \(t \geq 0\).

   d. Determine the times when the velocity is 0 over one period.

   e. Find the acceleration function.
f. Sketch one period of the acceleration function for \( t \geq 0 \).

29) The number of hamburgers sold at a fast-food restaurant in Pasadena, California, is given by \( y = 10 + 5\sin x \) where \( y \) is the number of hamburgers sold and \( x \) represents the number of hours after the restaurant opened at 11 a.m. until 11 p.m., when the store closes. Find \( y' \) and determine the intervals where the number of burgers being sold is increasing.

Solution: \( y' = 5\cos x \), increasing on \( (0, \frac{\pi}{2}), (\frac{3\pi}{2}, \frac{5\pi}{2}) \), and \( (\frac{7\pi}{2}, 12) \)

30) [T] The amount of rainfall per month in Phoenix, Arizona, can be approximated by \( y(t) = 0.5 + 0.3\cos t \), where \( t \) is months since January. Find \( y' \) and use a calculator to determine the intervals where the amount of rain falling is decreasing.

For the following exercises, use the quotient rule to derive the given equations.

31) \( \frac{d}{dx}(\cot x) = -\csc^2 x \)

32) \( \frac{d}{dx}(\sec x) = \sec x \tan x \)

33) \( \frac{d}{dx}(\csc x) = -\csc x \cot x \)

Use the definition of derivative and the identity

Solution: \( \cos(x+h) = \cos x \cosh - \sin x \sinh \) to prove that \( \frac{d}{dx}(\cos x) \) \( \frac{dx}{dx} = -\sin x \).

For the following exercises, find the requested higher-order derivative for the given functions.

34) \( \frac{d^3 y}{dx^3} \) of \( y = 3\cos x \)

Solution: \( 3\sin x \)

35) \( \frac{d^2 y}{dx^2} \) of \( y = 3\sin x + x^2 \cos x \)

36) \( \frac{d^4 y}{dx^4} \) of \( y = 5\cos x \)

Solution: \( 5\cos x \)

37) \( \frac{d^2 y}{dx^2} \) of \( y = \sec x + \cot x \)

38) \( \frac{d^3 y}{dx^3} \) of \( y = x^{10} - \sec x \)

Solution: \( 720x^7 - 5\tan(x)\sec^3(x) - \tan^3(x)\sec(x) \)

3.6: The Chain Rule

For the following exercises, given \( y = f(u) \) and \( u = g(x) \), find \( dy \) by using Leibniz’s notation for the chain rule:

\( \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \)
1) $y = 3u - 6, u = 2x^2$

2) $y = 6u^3, u = 7x - 4$

Solution: $(18u^2)^2 = 18(7x - 4)^2$

3) $y = \sin u, u = x - 1$

4) $y = \cos u, u = \frac{-x}{8}$

Solution: $-\sin u \cdot \frac{-1}{8} = -\sin \left(\frac{-x}{8}\right) \cdot \frac{-1}{8}$

5) $y = \tan u, u = 9x + 2$

6) $y = \sqrt{4u + 3}, u = x^2 - 6x$

Solution: $\frac{8x - 24}{2\sqrt{4u + 3}} = \frac{4x - 12}{\sqrt{4x^2 - 24x + 3}}$

For each of the following exercises,

a. decompose each function in the form $y = f(u)$ and $u = g(x)$, and

b. find $\frac{dy}{dx}$ as a function of $x$.

7) $y = (3x - 2)^6$

8) $y = (3x^2 + 1)^3$

Solution: a. $u = 3x^2 + 1$; b. $18x(3x^2 + 1)^2$}

9) $y = \sin^5(x)$

10) $y = \left(\frac{x}{7} + \frac{7}{x}\right)^7$

Solution: a. $f(u) = u^7, u = \frac{x}{7} + \frac{7}{x}$; b. $7(\frac{x}{7} + \frac{7}{x})^6 \cdot \frac{1}{7} - \frac{7}{x^2}$

11) $y = \tan(\sec x)$

12) $y = \csc(\pi x + 1)$

Solution: a. $f(u) = \csc u, u = \pi x + 1$; b. $-\csc(\pi x + 1) \cdot \cot(\pi x + 1)$

13) $y = \cot^2 x$

14) $y = -6\sin^{-3}x$
For the following exercises, find \( \frac{dy}{dx} \) for each function.

15) \( y = (3x^2 + 3x - 1)^4 \)

16) \( y = (5 - 2x)^{-2} \)

Solution: \( \frac{4}{(5 - 2x)^3} \)

17) \( y = \cos^3(\pi x) \)

18) \( y = (2x^3 - x^2 + 6x + 1)^3 \)

Solution: \( (6(2x^3 - x^2 + 6x + 1)^2(3x^2 - x + 3)) \)

19) \( y = \frac{1}{\sin^2(x)} \)

20) \( y = (\tan x + \sin x)^{-3} \)

Solution: \( -3(\tan x + \sin x)^{-4}(\sec^2 x + \cos x) \)

21) \( y = x^2 \cos^4 x \)

22) \( y = \sin(\cos 7x) \)

Solution: \( -7\cos(\cos 7x)\sin 7x \)

23) \( y = \sqrt{6 + \sec \pi x^2} \)

24) \( y = \cot^3(4x + 1) \)

Solution: \( -12\cot^2(4x + 1)\cot 2(4x + 1) \)

25) Let \( y = [f(x)]^3 \) and suppose that \( f'(1) = 4 \) and \( \frac{dy}{dx} = 10 \) for \( x = 1 \). Find \( f(1) \).

26) Let \( y = (f(x) + 5x^2)^4 \) and suppose that \( f(-1) = -4 \) and \( \frac{dy}{dx} = 3 \) when \( x = -1 \). Find \( f'(-1) \)

Solution: \( 10f'(4) \)

27) Let \( y = (f(u) + 3x)^2 \) and \( u = x^3 - 2x \). If \( f'(4) = 6 \) and \( \frac{dy}{dx} = 18 \) when \( x = 2 \), find \( f'(4) \).

28) \[ T \] Find the equation of the tangent line to \( y = -\sin(\frac{x}{2}) \) at the origin. Use a calculator to graph the function and the tangent line together.

Solution: \( y = \frac{-1}{2}x \)
29) [T] Find the equation of the tangent line to \(y=(3x+\frac{1}{x})^2\) at the point \((1,16)\). Use a calculator to graph the function and the tangent line together.

30) Find the \((x)\)-coordinates at which the tangent line to \(y=(x-\frac{6}{x})^8\) is horizontal.

Solution: \((x=±\sqrt{6})\)

31) [T] Find an equation of the line that is normal to \(g(θ)=\sin 2(πθ)\) at the point \((\frac{1}{4},\frac{1}{2})\). Use a calculator to graph the function and the normal line together.

For the following exercises, use the information in the following table to find \(h′(a)\) at the given value for \(a\).

<table>
<thead>
<tr>
<th>((x))</th>
<th>((f(x)))</th>
<th>((f'(x)))</th>
<th>((g(x)))</th>
<th>((g'(x)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>5</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>−2</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>−1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>−3</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

32) \(h(x)=f(g(x));a=0\)

Solution: \(10\)

33) \(h(x)=g(f(x));a=0\)

34) \(h(x)=(x^4+g(x))^{-2};a=1\)

Solution: \(−\frac{1}{8}\)

35) \(h(x)=(\frac{f(x)}{g(x)})^2;a=3\)

36) \(h(x)=f(x+f(x));a=1\)

Solution: \(−4\)

37) \(h(x)=(1+g(x))^3;a=2\)

38) \(h(x)=g(2+f(x^2));a=1\)

Solution: \(−12\)

39) \(h(x)=f(g(\sin x));a=0\)
40) [T] The position function of a freight train is given by
\[ s(t)=100(t+1)^{-2}, \]
with \( s \) in meters and \( t \) in seconds. At time \( t=6 \) s, find the train’s
a. velocity and
b. acceleration.

Using a. and b. is the train speeding up or slowing down?

Solution: a. \(-\frac{200}{343}\) m/s, b. \(\frac{600}{2401}\) m/s\(^2\). The train is slowing down since velocity and acceleration have opposite signs.

41) [T] A mass hanging from a vertical spring is in simple harmonic motion as given by the following position function, where \( t \) is measured in seconds and \( s \) is in inches:
\[ s(t)=-3\cos(\pi t+\frac{\pi}{4}). \]
a. Determine the position of the spring at \( t=1.5 \) s.
b. Find the velocity of the spring at \( t=1.5 \) s.

42) [T] The total cost to produce \( x \) boxes of Thin Mint Girl Scout cookies is \( C \) dollars, where
\[ C=0.0001x^3-0.02x^2+3x+300. \]
In \( t \) weeks production is estimated to be \( x=1600+100t \) boxes.

a. Find the marginal cost \( C'(x) \).
b. Use Leibniz’s notation for the chain rule, \( \frac{dC}{dt}=\frac{dC}{dx}\frac{dx}{dt} \), to find the rate with respect to time \( t \) that the cost is changing.
c. Use b. to determine how fast costs are increasing when \( t=2 \) weeks. Include units with the answer.

Solution: a. \( C'(x)=0.0003x^2-0.04x+3 \)
b. \( \frac{dC}{dt}=100(0.0003x^2-0.04x+3) \) c. Approximately $90,300 per week

43) [T] The formula for the area of a circle is \( A=\pi r^2 \), where \( r \) is the radius of the circle. Suppose a circle is expanding, meaning that both the area \( A \) and the radius \( r \) (in inches) are expanding.

a. Suppose \( r=2-(\frac{100}{(t+7)^2}) \) where \( t \) is time in seconds. Use the chain rule \( \frac{dA}{dt}=\frac{dA}{dr}\frac{dr}{dt} \) to find the rate at which the area is expanding.
b. Use a. to find the rate at which the area is expanding at \( t=4 \) s.

44) [T] The formula for the volume of a sphere is \( V=\frac{4}{3}\pi r^3 \), where \( r \) (in feet) is the radius of the sphere.
Suppose a spherical snowball is melting in the sun.

a. Suppose \( r = \frac{1}{(t+1)^2} - \frac{12}{t+1} \) where \( t \) is time in minutes. Use the chain rule \( \frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt} \) to find the rate at which the snowball is melting.

b. Use a. to find the rate at which the volume is changing at \( t = 1 \) min.

Solution: \( a. \frac{dS}{dt} = -\frac{8\pi r^2}{(t+1)^3} \) b. The volume is decreasing at a rate of \( -\frac{\pi}{36} \text{ ft}^3/\text{min} \)

45) [T] The daily temperature in degrees Fahrenheit of Phoenix in the summer can be modeled by the function \( T(x) = 94 - 10\cos\left(\frac{\pi}{12}(x-2)\right) \), where \( x \) is hours after midnight. Find the rate at which the temperature is changing at 4 p.m.

46) [T] The depth (in feet) of water at a dock changes with the rise and fall of tides. The depth is modeled by the function \( D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8 \), where \( t \) is the number of hours after midnight. Find the rate at which the depth is changing at 6 a.m.

Solution: \( -2.3 \text{ ft/hr} \)

### 3.7: Derivatives of Inverse Functions

For the following exercises, use the graph of \( y = f(x) \) to

a. sketch the graph of \( y = f^{-1}(x) \), and

b. use part a. to estimate \( (f^{-1})'(1) \).

1)
Solution:

a.

b. \((f^{-1})'(1) = 2\)

3)

b. \((f^{-1})(1) = -2\)
Solution:

a.
For the following exercises, use the functions \( y = f(x) \) to find

a. \( \frac{df}{dx} \) at \( x = a \) and

b. \( x = f^{-1}(y) \).

c. Then use part b. to find \( \frac{df^{-1}}{dy} \) at \( y = f(a) \).

5) \( f(x) = 6x - 1, x = -2 \)

6) \( f(x) = 2x^3 - 3, x = 1 \)

Solution: \( a. 6, b. x = f^{-1}(y) = (\frac{y + 3}{2})^{1/3}, c. \frac{1}{6} \)

7) \( f(x) = 9 - x^2, 0 \leq x \leq 3, x = 2 \)

8) \( f(x) = \sin x, x = 0 \)

Solution: \( a. 1, b. x = f^{-1}(y) = \sin^{-1}y, c. 1 \)

For each of the following functions, find \( ((f^{-1})')'(a)) \).

9) \( f(x) = x^2 + 3x + 2, x \geq -1, a = 2 \)

10) \( f(x) = x^3 + 2x + 3, a = 0 \)

Solution: \( \left( \frac{\text{df}}{\text{dy}} \right) \left( f^{-1}(y) \right) = \left( \frac{\text{df}}{\text{dx}} \right)^{-1} \left( f^{-1}(x) \right) \left( \frac{dx}{dy} \right) \)

\( b. ((f^{-1})')'(1) = -1/\sqrt{3} \)
11) \( f(x) = x + \sqrt{x}, a = 2 \)

12) \( f(x) = x - \frac{2}{x}, x < 0, a = 1 \)

Solution: \( \frac{1}{3} \)

13) \( f(x) = x + \sin x, a = 0 \)

14) \( f(x) = \tan x + 3x^2, a = 0 \)

Solution: \( 1 \)

For each of the given functions \( y = f(x) \),

a. find the slope of the tangent line to its inverse function \( f^{-1} \) at the indicated point \( P \), and

b. find the equation of the tangent line to the graph of \( f^{-1} \) at the indicated point.

15) \( f(x) = \frac{4}{1 + x^2}, P(2, 1) \)

16) \( f(x) = \sqrt{x - 4}, P(2, 8) \)

Solution: \( a. 4, b. y = 4x \)

17) \( f(x) = (x^3 + 1)^4, P(16, 1) \)

18) \( f(x) = -x^3 - x + 2, P(-8, 2) \)

Solution: \( a. -\frac{1}{96}, b. y = -\frac{1}{13}x + \frac{18}{13} \)

19) \( f(x) = x^5 + 3x^3 - 4x - 8, P(-8, 1) \)

For the following exercises, find \( \frac{dy}{dx} \) for the given function.

20) \( y = \sin^{-1}(x^2) \)

Solution: \( \frac{2x}{\sqrt{1 - x^4}} \)

21) \( y = \cos^{-1}(\sqrt{x}) \)

22) \( y = \sec^{-1}\left(\frac{1}{x}\right) \)

Solution: \( -\frac{1}{\sqrt{1 - x^2}} \)

23) \( y = \sqrt{\csc^{-1}x} \)
24) \( y = (1 + \tan^{-1} x)^{3} \)

Solution: \( \frac{3(1 + \tan^{-1} x)^{2}}{1 + x^{2}} \)

25) \( y = \cos^{-1}(-1)(2x)\sin^{-1}(-1)(2x) \)

26) \( y = \frac{1}{\tan^{-1}(-1)(x)} \)

Solution: \( \frac{-1}{(1 + x^{2})(\tan^{-1} x)^{2}} \)

27) \( y = \sec^{-1}(-1)(-x) \)

28) \( y = \cot^{-1}(-1)\sqrt{4-x^{2}} \)

Solution: \( \frac{x}{(5-x^{2})\sqrt{4-x^{2}}} \)

29) \( y = x?\csc^{-1}x \)

For the following exercises, use the given values to find \((f^{-1})'(a)\).

30) \( f(\pi) = 0, f'(\pi) = -1, a = 0 \)

Solution: \(-1\)

31) \( f(6) = 2, f'(6) = \frac{1}{3}, a = 2 \)

32) \( f'(\frac{1}{3}) = -8, f'(\frac{1}{3}) = 2, a = -8 \)

Solution: \(\frac{1}{2}\)

33) \( f(\sqrt{3}) = \frac{1}{2}, f'(-1)(\sqrt{3}) = \frac{2}{3}, a = \frac{1}{2} \)

34) \( f(1) = -3, f'(1) = 10, a = -3 \)

Solution: \(\frac{1}{10}\)

35) \( f(1) = 0, f'(1) = -2, a = 0 \)

36) \( T \) The position of a moving hockey puck after \( t \) seconds is \( s(t) = \tan^{-1} t \) where \( s \) is in meters.

   a. Find the velocity of the hockey puck at any time \( t \).

   b. Find the acceleration of the puck at any time \( t \).

   c. Evaluate a. and b. for \( t = 2, 4 \), and \( 6 \) seconds.
d. What conclusion can be drawn from the results in c.?

Solution: \(a. \ v(t) = \frac{1}{1+t^2}\ b. \ a(t) = -\frac{2t}{(1+t^2)^2}\ c. \ (a) 0.2, 0.06, 0.03; \ (b) -0.16, -0.028, -0.0088\)

The hockey puck is decelerating/slowing down at 2, 4, and 6 seconds.

37) [T] A building that is 225 feet tall casts a shadow of various lengths \(x\) as the day goes by. An angle of elevation \(\theta\) is formed by lines from the top and bottom of the building to the tip of the shadow, as seen in the following figure. Find the rate of change of the angle of elevation \(\frac{d\theta}{dx}\) when \(x=272\) feet.

![Diagram of building casting a shadow](image)

38) [T] A pole stands 75 feet tall. An angle \(\theta\) is formed when wires of various lengths of \(x\) feet are attached from the ground to the top of the pole, as shown in the following figure. Find the rate of change of the angle \(\frac{d\theta}{dx}\) when a wire of length 90 feet is attached.

![Diagram of pole with wires](image)

Solution: \((-0.0168\) radians per foot\)
39) [T] A television camera at ground level is 2000 feet away from the launching pad of a space rocket that is set to take off vertically, as seen in the following figure. The angle of elevation of the camera can be found by 
\( \theta = \tan^{-1}\left(\frac{x}{2000}\right) \), where \( x \) is the height of the rocket. Find the rate of change of the angle of elevation after launch when the camera and the rocket are 5000 feet apart.

40) [T] A local movie theater with a 30-foot-high screen that is 10 feet above a person’s eye level when seated has a viewing angle \( \theta \) (in radians) given by 
\( \theta = \cot^{-1}\left(\frac{x}{40}\right) - \cot^{-1}\left(\frac{x}{10}\right) \),

where \( x \) is the distance in feet away from the movie screen that the person is sitting, as shown in the following figure.

a. Find \( \frac{d\theta}{dx} \).

b. Evaluate \( \frac{d\theta}{dx} \) for \( x = 5, 10, 15, 20 \).

c. Interpret the results in b..
d. Evaluate $\frac{dθ}{dx}$ for $x=25,30,35,$ and 40

e. Interpret the results in d. At what distance $x$ should the person stand to maximize his or her viewing angle?

Solution: a. $\frac{dθ}{dx}=\frac{10}{100+x^2}−\frac{40}{1600+x^2}$

b. $\frac{18}{325}, \frac{9}{340}, \frac{42}{4745}, 0$

c. As a person moves farther away from the screen, the viewing angle is increasing, which implies that as he or she moves farther away, his or her screen vision is widening.
d. $\frac{-54}{12905}, \frac{-3}{500}, \frac{-198}{29945}, \frac{-9}{1360}$
e. As the person moves beyond 20 feet from the screen, the viewing angle is decreasing. The optimal distance the person should stand for maximizing the viewing angle is 20 feet.

3.8: Implicit Differentiation

For the following exercises, use implicit differentiation to find $\frac{dy}{dx}$.

1) $x^2−y^2=4$

2) $6x^2+3y^2=12$

Solution: $\frac{dy}{dx}=-\frac{2x}{y}$

3) $x^2y=y−7$

4) $3x^3+9xy^2=5x^3$

Solution: $\frac{dy}{dx}=-\frac{3x^2y−y^3}{x^3+3xy^2}$

5) $xy−cos(xy)=1$

6) $y\sqrt{x+4}=xy+8$

$\frac{dy}{dx}=-\frac{y−\frac{y}{2\sqrt{x+4}}}{\sqrt{x+4}−x}$

7) $−xy−2=\sqrt{x+4}$

8) $ysin(xy)=y^2+2$

Solution: $\frac{dy}{dx}=\frac{y^2cos(xy)}{2y−sin(xy)−xycosxy}$

9) $(xy)^2+3x=y^2$

10) $(x^3+y^3)^2=−8$

Solution: $\frac{dy}{dx}=\frac{-3x^2y−y^3}{x^3+3xy^2}$
For the following exercises, find the equation of the tangent line to the graph of the given equation at the indicated point. Use a calculator or computer software to graph the function and the tangent line.

11) \(x^4y−xy^3=−2,(−1,−1)\)

12) \(x^2y^2+5xy=14,(2,1)\)

Solution: \(y=−\frac{1}{2}x+2\)

13) \(\tan(xy)=y,\left(\frac{\pi}{4},1\right)\)

14) \(xy^2+\sin(\pi y)=10,(2,−3)\)

Solution: \(y=\frac{1}{\pi+12}x−\frac{3\pi+38}{\pi+12}\)
15) \( \frac{x}{y} + 5x - 7 = -\frac{3}{4}y, (1, 2) \)

16) \( xy + \sin(x) = 1, (\frac{\pi}{2}, 0) \)

Solution: \( y = 0 \)

17) The graph of a folium of Descartes with equation \( 2x^3 + 2y^3 - 9xy = 0 \) is given in the following graph.

a. Find the equation of the tangent line at the point \( (2, 1) \). Graph the tangent line along with the folium.

b. Find the equation of the normal line to the tangent line in a. at the point \( (2, 1) \).

18) For the equation \( x^2 + 2xy - 3y^2 = 0 \)
a. Find the equation of the normal to the tangent line at the point \((1,1)\).

b. At what other point does the normal line in a. intersect the graph of the equation?

Solution: \(a. y=x+2\ b. (3,−1)\)

19) Find all points on the graph of \((y^3−27y=x^2−90)\) at which the tangent line is vertical.

For the equation \((x^2+xy+y^2=7)\),

a. Find the \((x)-intercept(s).\)

b. Find the slope of the tangent line(s) at the \(x\)-intercept(s).

c. What does the value(s) in b. indicate about the tangent line(s)?

Solution: \(a. (±7√,0)\ b. −2\) c. They are parallel since the slope is the same at both intercepts.

19) Find the equation of the tangent line to the graph of the equation \((\sin^\{-1\}x+\sin^\{-1\}y=\frac{\pi}{6})\) at the point \((0,\frac{\pi}{2})\).

20) Find the equation of the tangent line to the graph of the equation \((\tan^\{-1\}(x+y)=x^2+\frac{\pi}{4})\) at the point \((0,1)\).

Solution: \((y=x+1)\)

21) Find \((y')\) and \((y'')\) for \((x^2+6xy−2y^2=3)\).

22) [T] The number of cell phones produced when \(x\) dollars is spent on labor and \(y\) dollars is spent on capital invested by a manufacturer can be modeled by the equation \((60x^{3/4}y^{1/4}=3240)\).

a. Find \((\frac{dy}{dx})\) and evaluate at the point \((81,16)\).

b. Interpret the result of a.

Solution: \((a. −0.5926)\ b. When$81 is spent on labor and $16 is spent on capital, the amount spent on capital is decreasing by $0.5926 per $1 spent on labor.

23) [T] The number of cars produced when \(x\) dollars is spent on labor and \(y\) dollars is spent on capital invested by a manufacturer can be modeled by the equation \((30x^{1/3}y^{2/3}=360)\).

(Both \((x)\) and \((y)\) are measured in thousands of dollars.)

a. Find \((\frac{dy}{dx})\) and evaluate at the point \((27,8)\).

b. Interpret the result of a.
24) The volume of a right circular cone of radius \(x\) and height \(y\) is given by \(V = \frac{1}{3} \pi x^2 y\). Suppose that the volume of the cone is \(85\pi \text{cm}^3\). Find \(\frac{dy}{dx}\) when \(x = 4\) and \(y = 16\).

Solution: \((-8)\)

25) For the following exercises, consider a closed rectangular box with a square base with side \(x\) and height \(y\).

Find an equation for the surface area of the rectangular box, \(S(x, y)\).

26) If the surface area of the rectangular box is 78 square feet, find \(\frac{dy}{dx}\) when \(x = 3\) feet and \(y = 5\) feet.

Solution: \((-2.67)\)

For the following exercises, use implicit differentiation to determine \(y'\). Does the answer agree with the formulas we have previously determined?

27) \((x = \sin y)\)

28) \((x = \cos y)\)

Solution: \((y' = -\frac{1}{\sqrt{1 - x^2}})\)

29) \((x = \tan y)\)

3.9: Derivatives of Exponential and Logarithmic Functions

For the following exercises, find \(f'(x)\) for each function.

1) \((f(x) = x^2 e^x)\)

Solution: \((2xe^x + x^2 e^x)\)

2) \((f(x) = \frac{e^{-x}}{x})\)

3) \((f(x) = e^{x^3 \ln x})\)

Solution: \((e^{x^3 \ln x}(3x^2 \ln x + x^2))\)

4) \((f(x) = \sqrt{e^{2x} + 2x})\)

5) \((f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}})\)

Solution: \((\frac{4}{(e^x + e^{-x})^2})\)

6) \((f(x) = \frac{10^x}{\ln 10})\)
7) \( f(x) = 2^{4x} + 4x^2 \)

Solution: \( (2^{4x+2}) ln(2) + 8x \)

8) \( f(x) = 3^{\sin(3x)} \)

9) \( f(x) = x^\pi \cdot \pi^x \)

Solution: \( (\pi x^{\pi-1}) \cdot \pi^x + x^\pi \cdot \pi^x \ln\pi \)

10) \( f(x) = \ln(4x^3 + x) \)

11) \( f(x) = \ln(\sqrt{5x - 7}) \)

Solution: \( (\frac{5}{2(5x - 7)}) \)

12) \( f(x) = x^2 \ln(9x) \)

13) \( f(x) = \log(\sec x) \)

Solution: \( (\frac{\tan x}{\ln 10}) \)

14) \( f(x) = \log_7(6x^4 + 3)^5 \)

15) \( f(x) = 2^x \cdot \log_3(7^{x^2 - 4}) \)

Solution: \( (2^x \cdot \ln 2 \cdot \log_3 7^{x^2 - 4} + 2^x \cdot \frac{2x \ln 7}{\ln 3}) \)

For the following exercises, use logarithmic differentiation to find \( \frac{dy}{dx} \).

16) \( y = x^{\sqrt{x}} \)

17) \( y = (\sin 2x)^{4x} \)

Solution: \((\sin 2x)^{4x} \cdot [4?\ln(\sin 2x) + 8x?\cot 2x]\)

18) \( y = (\ln x)^{\ln x} \)

19) \( y = x^{\log_2 x} \)

Solution: \((x^{\log_2 x} \cdot \frac{2 \ln x}{x \ln 2}) \)

20) \( y = (x^2 - 1)^{\ln x} \)

21) \( y = x^{\cot x} \)
Solution: \(x^{\cot x} \frac{\csc^2 x \ln x + \cot x}{x}\)

22) \(y = \frac{x + 11}{\sqrt[3]{x^2 - 4}}\)

23) \(y = x^{-1/2} (x^2 + 3)^{2/3} (3x - 4)^4\)

Solution: \(x^{-1/2} (x^2 + 3)^{2/3} (3x - 4)^4 \frac{-1}{2x} + \frac{4x}{3(x^2 + 3)} + \frac{12}{3x - 4}\)

24) \([T]\) Find an equation of the tangent line to the graph of \(f(x) = 4x e^{(x^2 - 1)}\) at the point where \(x = -1\). Graph both the function and the tangent line.

25) \([T]\) Find the equation of the line that is normal to the graph of \(f(x) = x^{5^x}\) at the point where \(x = 1\). Graph both the function and the normal line.

Solution: \(y = \frac{-1}{5 + 5 \ln 5} x + (5 + \frac{1}{5 + 5 \ln 5})\)

26) \([T]\) Find the equation of the tangent line to the graph of \((x^3 - x \ln y + y^3 = 2x + 5)\) at the point where \(x = 2\). (Hint: Use implicit differentiation to find \(\frac{dy}{dx}\).) Graph both the curve and the tangent line.

27) Consider the function \(y = x^{1/x}\) for \(x > 0\).

   a. Determine the points on the graph where the tangent line is horizontal.

   b. Determine the points on the graph where \(y' > 0\) and those where \(y' < 0\).

Solution: \(a. x = e^{-2.718}\) \(b. (e, \infty), (0, e)\)

28) The formula \(I(t) = \frac{\sin t}{e^t}\) is the formula for a decaying alternating current.
a. Complete the following table with the appropriate values.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\frac{\sin t}{e^t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(i)</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>(ii)</td>
</tr>
<tr>
<td>$\pi$</td>
<td>(iii)</td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>(vi)</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>(v)</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>(vi)</td>
</tr>
<tr>
<td>$3\pi$</td>
<td>(vii)</td>
</tr>
</tbody>
</table>

b. Using only the values in the table, determine where the tangent line to the graph of $f(t)$ is horizontal.

29) [T] The population of Toledo, Ohio, in 2000 was approximately 500,000. Assume the population is increasing at a rate of 5% per year.

   a. Write the exponential function that relates the total population as a function of $t$.

   Solution: $P = 500,000(1.05)^t$ individuals

   b. Use a. to determine the rate at which the population is increasing in $t$ years.

   c. Use b. to determine the rate at which the population is increasing in 10 years

   Solution: $P'(t) = 24395(1.05)^t$ individuals per year

   $39,737$ individuals per year

30) [T] An isotope of the element erbium has a half-life of approximately 12 hours. Initially there are 9 grams of the isotope present.

   a. Write the exponential function that relates the amount of substance remaining as a function of $t$, measured in hours.

   b. Use a. to determine the rate at which the substance is decaying in $t$ hours.

   c. Use b. to determine the rate of decay at $t=4$ hours.

31) [T] The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1961 is modeled by the function

   $N(t) = 5.3e^{0.093t^2 - 0.87t}$, $(0 \leq t \leq 4)$,

where $N(t)$ gives the number of cases (in thousands) and $t$ is measured in years, with $t=0$ corresponding to the beginning of 1960.
a. Show work that evaluates \(N(0)\) and \(N(4)\). Briefly describe what these values indicate about the disease in New York City.

b. Show work that evaluates \(N'(0)\) and \(N'(3)\). Briefly describe what these values indicate about the disease in the United States.

a. At the beginning of 1960 there were 5.3 thousand cases of the disease in New York City. At the beginning of 1963 there were approximately 723 cases of the disease in the United States. b. At the beginning of 1960 the number of cases of the disease was decreasing at rate of \((-4.611\) thousand per year; at the beginning of 1963, the number of cases of the disease was decreasing at a rate of \((-0.2808\) thousand per year.

32) [T] The relative rate of change of a differentiable function \(y=f(x)\) is given by \(\frac{100\cdot f'(x)}{f(x)}\%).\) One model for population growth is a Gompertz growth function, given by \(P(x)=ae^{-b\cdot e^{-cx}}\) where \(a, b,\) and \(c\) are constants.

a. Find the relative rate of change formula for the generic Gompertz function.

b. Use a. to find the relative rate of change of a population in \(x=20\) months when \(a=204, b=0.0198,\) and \(c=0.15.\)

c. Briefly interpret what the result of b. means.

33) For the following exercises, use the population of New York City from 1790 to 1860, given in the following table.

<table>
<thead>
<tr>
<th>Year since 1790</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>33,131</td>
</tr>
<tr>
<td>10</td>
<td>60,515</td>
</tr>
<tr>
<td>20</td>
<td>96,373</td>
</tr>
<tr>
<td>30</td>
<td>123,706</td>
</tr>
<tr>
<td>40</td>
<td>202,300</td>
</tr>
<tr>
<td>50</td>
<td>312,710</td>
</tr>
<tr>
<td>60</td>
<td>515,547</td>
</tr>
<tr>
<td>70</td>
<td>813,669</td>
</tr>
</tbody>
</table>

New York City Population Over TimeSource: [http://en.wikipedia.org/wiki/Largest_cities_in_the_United_States_by_population_by_decade](http://en.wikipedia.org/wiki/Largest_cities_in_the_United_States_by_population_by_decade)

34) [T] Using a computer program or a calculator, fit a growth curve to the data of the form \(p=ab^t.\)
Solution: \( p = 35741 (1.045)^t \)

35) [T] Using the exponential best fit for the data, write a table containing the derivatives evaluated at each year.

36) [T] Using the exponential best fit for the data, write a table containing the second derivatives evaluated at each year.

Solution:

<table>
<thead>
<tr>
<th>Year since 1790</th>
<th>( P'' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>69.25</td>
</tr>
<tr>
<td>10</td>
<td>107.5</td>
</tr>
<tr>
<td>20</td>
<td>167.0</td>
</tr>
<tr>
<td>30</td>
<td>259.4</td>
</tr>
<tr>
<td>40</td>
<td>402.8</td>
</tr>
<tr>
<td>50</td>
<td>625.5</td>
</tr>
<tr>
<td>60</td>
<td>971.4</td>
</tr>
<tr>
<td>70</td>
<td>1508.5</td>
</tr>
</tbody>
</table>

37) [T] Using the tables of first and second derivatives and the best fit, answer the following questions:

a. Will the model be accurate in predicting the future population of New York City? Why or why not?

b. Estimate the population in 2010. Was the prediction correct from a.?