11.2: Arc Length

In Section 4.1 we saw that one revolution has a radian measure of \(2\pi\) rad. Note that \(2\pi\) is the ratio of the circumference (i.e. total arc length) \(C\) of a circle to its radius \(r\):

\[
\text{Radian measure of 1 revolution} \approx 2\pi \approx \frac{2\pi \cdot r}{r} \approx \frac{C}{r} \approx \frac{\text{total arc length}}{\text{radius}}
\]

Clearly, that ratio is independent of \(r\). In general, the radian measure of an angle is the ratio of the arc length cut off by the corresponding central angle in a circle to the radius of the circle, independent of the radius.

Figure 4.2.1 Radian measure and arc length

Now suppose that we cut the angle with radian measure \(1\) in half, as in Figure 4.2.1(b). Clearly, this cuts the arc length \(r\) in half as well. Thus, we see that

\[
\begin{align*}
\text{Angle} &= 1 \text{ radian} & \Rightarrow & & \text{arc length} &= r \\
\text{Angle} &= 2 \text{ radians} & \Rightarrow & & \text{arc length} &= 2r
\end{align*}
\]
\[ \text{Angle} \quad \Rightarrow \quad \text{arc length} \quad \Rightarrow \quad \theta \quad \text{radians} \quad \Rightarrow \quad \theta \, r \quad \Rightarrow \quad \theta = \frac{\text{arc length}}{\text{radius}} \]

Intuitively, it is obvious that shrinking or magnifying a circle preserves the measure of a central angle even as the radius changes. The above discussion says more, namely that the ratio of the length \( s \) of an intercepted arc to the radius \( r \) is preserved, precisely because that ratio is the measure of the central angle in radians (see Figure 4.2.2).

![Figure 4.2.2 Circles with the same central angle, different radii](image)

We thus get a simple formula for the length of an arc:

In a circle of radius \( r \), let \( s \) be the length of an arc intercepted by a central angle with radian measure \( \theta \ge 0 \). Then the arc length \( s \) is:

\[ s = r \theta \]

So first convert \( \theta = 41^\circ \) to radians, then use \( s = r \theta \):

\[ \theta = 41^\circ \quad \Rightarrow \quad \frac{\pi}{180} \cdot 41 \approx 0.716 \quad \Rightarrow \quad s = r \theta \approx (10\, \text{ft})(0.716) \approx 7.16\, \text{ft} \]

Note that since the arc length \( s \) and radius \( r \) are usually given in the same units, radian measure is really unitless, since you can think of the units canceling in the ratio \( \frac{s}{r} \), which is just \( \theta \). This is another reason why radians are so widely used.

For central angles \( \theta > 2\pi \) rad, i.e. \( \theta > 360^\circ \), it may not be clear what is meant by the intercepted arc,
since the angle is larger than one revolution and hence "wraps around" the circle more than once. We will take the approach that such an arc consists of the full circumference plus any additional arc length determined by the angle. In other words, Equation \ref{4.4} is still valid for angles $\theta > 2\pi$ rad.

What about negative angles? In this case using $s=r\theta$ would mean that the arc length is negative, which violates the usual concept of length. So we will adopt the convention of only using nonnegative central angles when discussing arc length.

A rope is fastened to a wall in two places $8$ ft apart at the same height. A cylindrical container with a radius of $2$ ft is pushed away from the wall as far as it can go while being held in by the rope, as in Figure 4.2.3 which shows the top view. If the center of the container is $3$ feet away from the point on the wall midway between the ends of the rope, what is the length $L$ of the rope?

\[ L = 2 \cdot (AB + \overparen{BC}) \]

![Figure 4.2.3](image.png)

**Solution:**

We see that, by symmetry, the total length of the rope is $L = 2 \cdot (AB + \overparen{BC})$. Also, notice that $\triangle ADE$ is a right triangle, so the hypotenuse has length $AE = \sqrt{DE^2 + DA^2} = \sqrt{3^2 + 4^2} = 5$ ft, by the Pythagorean Theorem. Now since $\overline{AB}$ is tangent to the circular container, we know that $\angle ABE$ is a right angle. So by the Pythagorean Theorem we have

\[ AB = \sqrt{AE^2 - BE^2} = \sqrt{5^2 - 2^2} = \sqrt{21} \text{ ft} \]

By Equation \ref{4.4} the arc $\overparen{BC}$ has length $BE \cdot \theta$, where $\theta = \angle BEC$ is the supplement of $\angle AED + \angle AEB$. So since

\[ \tan \angle AED = \frac{4}{3} \Rightarrow \angle AED = 53.1^\circ \quad \text{and} \quad \angle AEB = 90^\circ \]

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\[ \cos \angle AEB \approx \frac{BE}{AE} \approx \frac{2}{5} \Rightarrow \angle AEB \approx 66.4^\circ, \]

we have

\[
\theta = \angle BEC \approx 180^\circ - (\angle AED + \angle AEB) \approx 180^\circ - (53.1^\circ + 66.4^\circ) \approx 60.5^\circ.
\]

Converting to radians, we get \( \theta = \frac{\pi}{180} \cdot 60.5 = 1.06 \) rad. Thus,

\[
L = 2 \cdot (AB + \cdot \overparen{BC}) = 2 \cdot (\sqrt{21} + BE \cdot \theta) = 2 \cdot (\sqrt{21} + (2) \cdot (1.06)) = \boxed{13.4 \text{ ft}}.
\]

Figure 4.2.4 Belt pulleys with radii 5 cm and 8 cm

First, at the center \( B \) of the pulley with radius \( 8 \), draw a circle of radius \( 3 \), which is the difference in the radii of the two pulleys. Let \( C \) be the point where this circle intersects \( \overline{BF} \). Then we know that the tangent line \( \overline{AC} \) to this smaller circle is perpendicular to the line segment \( \overline{BF} \). Thus, \( \angle ACB \) is a right angle, and so the length of \( \overline{AC} \) is

\[
AC \approx \sqrt{AB^2 - BC^2} \approx \sqrt{15^2 - 3^2} \approx \sqrt{216} \approx 6\sqrt{6}
\]

by the Pythagorean Theorem. Now since \( \overline{AE} \perp \overline{EF} \) and \( \overline{EF} \perp \overline{CF} \) and \( \overline{CF} \perp \overline{AC} \), the quadrilateral \( \overline{AEFC} \) must be a rectangle. In particular, \( \angle ACB \) is a right angle, and so the length of \( \overline{AC} \) is

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By Equation \ref{4.4} we know that \( \overparen{DE} = EA \cdot \angle DAE \) and \( \overparen{FG} = BF \cdot \angle GBF \), where the angles are measured in radians. So thinking of angles in radians (using \( \pi \text{ rad} \approx 180^\circ \)), we see from Figure 4.2.4 that
\[ \angle DAE = \pi - \angle EAC - \angle BAC = \pi - \frac{\pi}{2} - \angle BAC, \]

where
\[ \sin \angle BAC = \frac{BC}{AB} = \frac{3}{15} = 0.2 \quad \Rightarrow \quad \angle BAC = 0.201 \text{ rad.} \]

Thus, \( \angle DAE = \frac{\pi}{2} - 0.201 = 1.37 \) rad. So since \( \overline{AE} \) and \( \overline{BF} \) are parallel, we have \( \angle ABC = \angle DAE = 1.37 \) rad. Thus, \( \angle GBF = \pi - \angle ABC = \pi - 1.37 = 1.77 \) rad. Hence,

\[ L = 2(\overparen{DE} + EF + \overparen{FG}) = 2(5(1.37) + 6\sqrt{6} + 8(1.77)) = \boxed{71.41 \text{ cm}}. \]

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