2.2: Classification of Differential Equations

Recall that a differential equation is an equation (has an equal sign) that involves derivatives. Just as biologists have a classification system for life, mathematicians have a classification system for differential equations. We can place all differential equation into two types: ordinary differential equation and partial differential equations.

- A **partial differential equation** is a differential equation that **involves** partial derivatives.
- An **ordinary differential equation** is a differential equation that **does not** involve partial derivatives.

Examples

\[
\frac{d^2y}{dx^2} + \frac{dy}{dx} = 3x \sin y
\]

is an ordinary differential equation since it does not contain partial derivatives. While

\[
\frac{\partial y}{\partial t} + x \frac{\partial y}{\partial x} = \frac{x+t}{x-t}
\]

is a partial differential equation, since \(y\) is a function of the two variables \(x\) and \(t\) and partial derivatives are present.

In this course we will focus on only ordinary differential equations.

**Order**

Another way of classifying differential equations is by order. Any ordinary differential equation can be written in the form

\[
F(x, y, y', y'', ..., y^{(n)}) = 0
\]
by setting everything equal to zero. The order of a differential equation is the **highest** derivative that appears in the above equation.

Examples

\[
\frac{d^2y}{dx^2} + \frac{dy}{dx} = 3x \sin \; y
\]

is a second order differential equation, since a second derivative appears in the equation.

\[
3y^4y''' - x^3y' + e^{xy}y = 0
\]

is a third order differential equation.

Once we have written a differential equation in the form

\[
F(x,y,y',y'',...,y^{(n)}) = 0
\]

we can talk about whether a differential equation is linear or not. We say that the differential equation above is a linear differential equation if

\[
\frac{\partial F}{\partial y^{(i)} \partial y^{(j)}} = 0
\]

for all \((i)\) and \((j)\). Any linear ordinary differential equation of degree \((n)\) can be written as

\[
a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = g(x).
\]

Examples

\[
3x^2y'' + 2\ln \; (x)y' + e^x \; y = 3x \cos \; x
\]

is a second order linear ordinary differential equation.

\[
4yy''' - x^3y' + \cos \; y = e^{2x}
\]

is **not** a linear differential equation because of the \((4yy''')\) and the \((\cos \; y)\) terms.

Nonlinear differential equations are often very difficult or impossible to solve. One approach getting around this difficulty is to **linearize** the differential equation.

Example: Linearization

\[
y'' + 2y' + e^y = x
\]

is nonlinear because of the \((e^y)\) term. However, the Taylor expansion of the exponential function

\[
e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \ldots
\]
can be approximated by the first two terms
\[ e^y \approx 1 + y. \]

We instead solve the much easier linear differential equation
\[ y'' + 2y' + 1 + y = x. \]

We say that a function \( f(x) \) is a solution to a differential equation if plugging in \( f(x) \) into the equation makes the equation equal.

Example \( \PageIndex{5} \)

Show that
\[ f(x) = x + e^{2x} \]
is a solution to
\[ y'' - 2y' = -2. \]

Solution

Taking derivatives:
\[ f'(x) = 1 + 2e^{2x} , f''(x) = 4e^{2x}. \]

Now plug in to get
\[ 4e^{2x} - 2(1 + 2e^{2x}) = 4e^{2x} - 2 - 4e^{2x} = -2. \]

Hence it is a solution.

Two questions that will be asking repeatedly of a differential equation course are

1. Does there exist a solution to the differential equation?
2. Is the solution given unique?

In the example above, the answer to the first question is yes since we verified that
\[ f(x) = x + e^{2x} \]
is a solution. However, the answer to the second question is no. It can be verified that
\[ s(x) = 4 + x \]
is also a solution.

Larry Green (Lake Tahoe Community College)

• Integrated by Justin Marshall.