2.2: Classification of Differential Equations

Recall that a differential equation is an equation (has an equal sign) that involves derivatives. Just as biologists have a classification system for life, mathematicians have a classification system for differential equations. We can place all differential equation into two types: ordinary differential equation and partial differential equations.

- A partial differential equation is a differential equation that involves partial derivatives.
- An ordinary differential equation is a differential equation that does not involve partial derivatives.

Examples

\[ \frac{d^2y}{dx^2} + \frac{dy}{dx} = 3x \sin y \]

is an ordinary differential equation since it does not contain partial derivatives. While

\[ \frac{\partial y}{\partial t} + x \frac{\partial y}{\partial x} = \frac{x+t}{x-t} \]

is a partial differential equation, since \(y\) is a function of the two variables \(x\) and \(t\) and partial derivatives are present.

In this course we will focus on only ordinary differential equations.

Order

Another way of classifying differential equations is by order. Any ordinary differential equation can be written in the form

\[ F(x,y,y',...,y^{(n)})=0 \]
by setting everything equal to zero. The order of a differential equation is the highest derivative that appears in the above equation.

Examples

\[
\frac{d^2y}{dx^2} + \frac{dy}{dx} = 3x \sin \ y
\]

is a second order differential equation, since a second derivative appears in the equation.

\[
3y^4 \frac{d^3y}{dx^3} - x^3 \frac{dy}{dx} + e^{xy}y = 0
\]

is a third order differential equation.

Once we have written a differential equation in the form

\[
F(x,y,y',...,y^{(n)}) = 0
\]

we can talk about whether a differential equation is linear or not. We say that the differential equation above is a linear differential equation if

\[
\frac{\partial F}{\partial y^{(i)} \partial y^{(j)}} = 0
\]

for all \(i\) and \(j\). Any linear ordinary differential equation of degree \(n\) can be written as

\[
a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = g(x).
\]

Examples

\[
3x^2y'' + 2\ln x y' + e^x y = 3x \cos x
\]

is a second order linear ordinary differential equation.

\[
4y'y'' - x^3 y' + \cos y = e^{2x}
\]

is not a linear differential equation because of the \(4y'y''\) and the \(\cos y\) terms.

Nonlinear differential equations are often very difficult or impossible to solve. One approach getting around this difficulty is to linearize the differential equation.

Example: Linearization

\[
y'' + 2y' + e^y = x
\]

is nonlinear because of the \(e^y\) term. However, the Taylor expansion of the exponential function

\[
e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \ldots
\]
can be approximated by the first two terms
\[ e^y \approx 1 + y. \]

We instead solve the much easier linear differential equation
\[ y'' + 2y' + 1 + y = x. \]

We say that a function \( f(x) \) is a solution to a differential equation if plugging in \( f(x) \) into the equation makes the equation equal.

Example \( \PageIndex{5} \)

Show that
\[ f(x) = x + e^{2x} \]
is a solution to
\[ y'' - 2y' = -2. \]

Solution

Taking derivatives:
\[ f'(x) = 1 + 2e^{2x}, \quad f''(x) = 4e^{2x}. \]

Now plug in to get
\[ 4e^{2x} - 2(1 + 2e^{2x}) = 4e^{2x} - 2 - 2e^{2x} = -2. \]

Hence it is a solution.

Two questions that will be asking repeatedly of a differential equation course are

1. Does there exist a solution to the differential equation?
2. Is the solution given unique?

In the example above, the answer to the first question is yes since we verified that
\[ f(x) = x + e^{2x} \]
is a solution. However, the answer to the second question is no. It can be verified that
\[ s(x) = 4 + x \]
is also a solution.

Larry Green (Lake Tahoe Community College)