1.2: Square Root Property, Complete the Square, and The Quadratic Formula

A quadratic equation in **standard form** is \(a x ^ { 2 } + b x + c = 0\) where \(a, b, c\) are real numbers and \(a ≠ 0\). Quadratic equations can have two real solutions, one real solution, or no real solution—in which case there will be two complex solutions.

Reviewing a few definitions pertaining to complex numbers:

- **A complex number** is a number in the form \(a+bi\) where \(a\) and \(b\) are real numbers. The real part is \(a\) and the imaginary part is \(b\).
- **The imaginary unit** is denoted as \(i\) and is defined as \(i=\sqrt{-1}\) and so \(i^2=-1\).
- **The square root of a negative number** should always be written in its complex number form: \(\sqrt{-b} = i\sqrt{b}\).

When it is factorable, a quadratic equation can be solved by factoring. In this section, an approach to solving the special case \((a x ^ { 2 } + c = 0)\) where there is no linear term (i.e. \(b=0\)), is examined. As an example, \((4x^2 - 9 = 0)\) can be solved by factoring as follows:
\[ 4 x^2 - 9 = 0 \]
\[ (2x + 3)(2x - 3) = 0 \]

\[ 2x + 3 = 0 \quad \text{or} \quad 2x - 3 = 0 \]
\[ 2x = -3 \quad \quad 2x = 3 \]
\[ x = -\frac{3}{2} \quad \quad x = \frac{3}{2} \]

The solution set is \( \{ \pm \frac{3}{2} \} \). Here we use \( \pm \) to write the two solutions in a more compact form.

An alternative method that can be used to more easily solve this equation comes from first isolating \( x^2 \) and then taking the square root of both sides of the equation. Besides its simplicity, this method allows us to solve equations that do not factor.

Continuing the above example to illustrate this alternative approach:

\[ 4 x^2 - 9 = 0 \]
\[ 4 x^2 = 9 \quad \text{First, isolate the squared term} \]
\[ x^2 = \frac{9}{4} \]
\[ \sqrt{x^2} = \pm \sqrt{\frac{9}{4}} \quad \text{Take the square root of both sides of the equal sign. Always remember the } \pm \text{ !!!} \]
\[ x = \pm \frac{3}{2} \quad \text{Simplify the radical} \]

In summary, when there is no linear term in a quadratic equation, one method to solve it is to use the square root property. In this approach, the \( x^2 \) term (or more generally the squared term) is isolated first, and then the square root of both sides of the equal sign is taken.

**THE SQUARE ROOT PROPERTY**

Given an algebraic expression \( u \), and a nonzero real number \( k \), then the equation \( u^2 = k \) has exactly two solutions.

If \( u^2 = k \), then \( u = \pm \sqrt{k} \) which is equivalent to \( u = \sqrt{k} \) and \( u = -\sqrt{k} \)

**How to: use the square root property to solve an equation**

This method can only be used on equations that have a squared expression and a constant term, but no linear term.

1. Isolate the squared expression \( u^2 \) on one side of the equal sign. Here \( u \) could simply be a single variable like \( x \), or it could be an expression involving \( x \).
2. Take the square root of both sides of the equation, putting a \( \pm \) sign before the expression on the side opposite the squared expression.
3. Simplify the numbers on the side with the \( \pm \) sign.

**Example \( \PageIndex{1} \):**

Solve: \( 9 x^2 - 8 = 0 \).

**Solution**

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\[
9x^2 - 8 = 0
\]
The Square Root Property can be used
\[
x^2 = \frac{8}{9}
\]
Isolate the squared term
\[
x = \pm \sqrt{\frac{8}{9}}
\]
Square root both sides. Remember the \pm sign!!!
\[
= \pm \frac{2 \sqrt{2}}{3}
\]
Simplify. Solution Set: \[\left\{ \pm \frac{2 \sqrt{2}}{3} \right\}\]

For completeness, these two solutions can be checked.

Check \(x = -\frac{2 \sqrt{2}}{3}\)
\[
9 \left(-\frac{2 \sqrt{2}}{3}\right)^2 - 8 = 0
\]
\[
9 \cdot \frac{8}{9} - 8 = 0
\]
\[
0 = 0 \quad \checkmark
\]

Check \(x = \frac{2 \sqrt{2}}{3}\)
\[
9 \left(\frac{2 \sqrt{2}}{3}\right)^2 - 8 = 0
\]
\[
9 \cdot \frac{8}{9} - 8 = 0
\]
\[
0 = 0 \quad \checkmark
\]

Sometimes quadratic equations have no real solution. In this case, the solutions will be complex numbers.

Example \(\PageIndex{2}\):

Solve: \(x^2 + 31 = 6\).

Solution
\[
\begin{aligned}
x^2 + 31 &= 6 \\
x &= \pm \sqrt{-25} \\
&= \pm 5i
\end{aligned}
\]

The Square Root Property can also be used on equations that have a squared expression rather than just a simple squared variable.

Example \(\PageIndex{3}\):

Solve \((x + 5)^2 = 9\).

Solution
\[
\begin{aligned}
(x + 5)^2 &= 9 \\
x + 5 &= \pm \sqrt{9} \\
&= \pm 3
\end{aligned}
\]
At this point, separate the “plus or minus” into two equations and solve each individually.

\[
\begin{aligned}
& x = - 5 + 3 \\
& x = - 5 - 3 \\
\end{aligned}
\]

In addition to fewer steps, this method allows us to solve equations that do not factor.

Example \(\PageIndex{4}\):

Solve: \((2 \ ( x - 2 ) \ ^\wedge\ 2 \ -\ 1\ =\ 4)\).

**Solution**

\[
\begin{aligned}
& 2 \ ( x - 2 ) \ ^\wedge\ 2 \ -\ 1\ =\ 4 \quad \text{The Square Root Property can be used} \\
& 2 \ ( x - 2 ) \ ^\wedge\ 2 \ =\ 5 \quad \text{Isolate the squared term} \\
& x - 2 \ =\ \pm \sqrt{\frac{5}{2}} \quad \text{Square root both sides. Remember the } \pm \quad \text{!!!} \\
& x \ =\ 2 \pm \frac{\sqrt{10}}{2} \quad \text{Simplify. } \end{aligned}
\]

Example \(\PageIndex{5}\): Solving a Quadratic Equation Using the Square Root Property

Solve \((25x^4-9=7)\).

**Solution**

First, isolate the \((x^2)\) term. Then take the square root of both sides.

\[
\begin{aligned}
& 25x^4-9\ =\ 7 \quad \text{The Square Root Property can be used} \\
& 25x^4\ =\ 16 \quad \text{Isolate the squared term} \\
& x^2\ =\ \pm \sqrt{\frac{4}{5}} \quad \text{Simplify. } \end{aligned}
\]
\[
x = \pm \frac{2}{\sqrt{5}} \quad \text{and} \quad x = \pm \sqrt{\frac{2i}{5}} \\
x = \pm \frac{2\sqrt{5}}{5} \quad \text{and} \quad x = \pm \frac{2i\sqrt{5}}{5} \quad \text{and} \\
\quad \textbf{Solution Set: } \left\{ \frac{2\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5}, \frac{2i\sqrt{5}}{5}, -\frac{2i\sqrt{5}}{5} \right\}
\]

Try It \(\PageIndex{6}\)

Solve.

a) \((2x^2 + 3 = 0)\)  \hspace{1cm} b) \((3(x-4)^2 = 15)\)  \hspace{1cm} c) \((2(3x - 1)^2 + 9 = 0)\)

\textbf{Answer}

a) \(\pm \frac{\sqrt{6}}{2} i\)  \hspace{1cm} b) \(x = 4 \pm \sqrt{5}\)  \hspace{1cm} c) \(\pm \sqrt{\frac{1}{3} \pm \frac{\sqrt{2}}{2} i}\)
Completing the Square

Not all quadratic equations can be factored or can be solved in their original form using the square root property. In these cases, we may use a method for solving a quadratic equation known as completing the square. Using this method, we add or subtract terms to both sides of the equation until we have a perfect square trinomial on one side of the equal sign. We then apply the square root property. To complete the square, the leading coefficient, \(a\), must equal \(1\). If it does not, then divide the entire equation by \(a\) before beginning the complete the square process. Then, we can use the following procedures to solve a quadratic equation by completing the square.

We will use the example \((x^2+6x+1=0)\) to illustrate each step. This is a quadratic equation that cannot be factored and with \((a=1)\).

\[
\begin{align*}
x^2+6x+1&= 0 && \text{Put the constant on the right of the equal sign; the } x \text{ and } x^2 \text{ terms on the other side}\\x^2+6x&= -1 && \text{Isolate the } x \text{ and } x^2 \text{ terms.}\x^2+6x + \Box &= -1 + \Box && \text{Add a constant } \Box \text{ to create a trinomial that is also a binomial square: } (x + \bigcirc)^2\x^2+6x + \Box &= -1 + \Box && \text{The binomial square constant, } \bigcirc, \text{ is always HALF the } x \text{ coefficient}\x^2+6x + \Box &= -1 + \Box && \text{The number added to both sides of the equal sign, } \Box \text{, is the square of that constant: } \Box^2 \x^2+6x + \Box^2 &= -1 + \Box && \text{The resulting trinomial will be } (x + \bigcirc)^2\x^2+6x + \Box^2 &= -1 + \Box && \text{The number added to both sides of the equal sign, } \Box \text{, is the square of that constant: } \Box^2 \x^2+6x + \Box^2 &= -1 + \Box && \text{Now use the square root property to solve the resulting equation.}\sqrt{(x+3)^2} &= \pm \sqrt{8} \x+3 &= \pm 2\sqrt{2} \x&= -3 \pm 2\sqrt{2} \textbf{ Solution Set: } \{-3+2\sqrt{2}, -3-2\sqrt{2}\}
\end{align*}
\]

Howto: Use Completing the Square to solve an equation

1. The coefficient of the \((x^2)\) term MUST be 1. If it is not 1, divide both sides of the equal sign by the coefficient of the \((x^2)\) term to make it 1.
2. Isolate the variable (\((x)\) and \((x^2)\)) terms on one side of the equal sign.
3. Add a constant to both sides of the equal sign that combined with the \((x)\) and \((x^2)\) terms will create a trinomial that is a perfect square binomial.
   1. Add the square of half the coefficient of \((x)\) to both sides of the equation.
   2. Factor the resulting trinomial into a binomial square.
4. Use the Square Root Property to solve for \((x)\). Remember the \((\pm)\) sign !!

This technique can be used to solve ANY quadratic equation, whereas factoring works only some of the time.
Example \(\PageIndex{7}\):

Solve by completing the square: \(x^2 - 8x - 2 = 0\).

**Solution**

\[
\begin{aligned}
    x^2 - 8x - 2 &= 0 \\
    x^2 - 8x &= 2 \\
    x^2 - 8x + 16 &= 2 + 16 \\
    (x-4)^2 &= 18 \\
    \sqrt{(x-4)^2} &= \pm \sqrt{18} \\
    x &= 4 \pm 3\sqrt{2}
\end{aligned}
\]

**Solution Set:** \(\{ 4+3\sqrt{2}, 4-3\sqrt{2} \}\)

Example \(\PageIndex{8}\):

Solve by completing the square: \(x^2 - 10x + 26 = 0\).

**Solution**

\[
\begin{aligned}
    x^2 - 10x + 26 &= 0 \\
    x^2 - 10x &= -26 \\
    x^2 - 10x + 25 &= -26 + 25 \\
    (x-5)^2 &= -1 \\
    \sqrt{(x-5)^2} &= \pm \sqrt{-1} \\
    x &= 5 \pm i
\end{aligned}
\]

**Solution Set:** \(\{ 5+i, 5-i \}\)

Example \(\PageIndex{9}\):

Solve by completing the square: \(x^2 + 2x - 48 = 0\).

**Solution**

\[
\begin{aligned}
    x^2 + 2x - 48 &= 0 \\
    x^2 + 2x &= 48 \\
    x^2 + 2x + 1 &= 48 + 1 \\
    (x+1)^2 &= 49 \\
    \sqrt{(x+1)^2} &= \pm \sqrt{49} \\
    x &= 5 \pm 7
\end{aligned}
\]
When the coefficient of \((x)\) is not divisible by \((2)\), the constant added in the complete the square process will be a fraction.

Example \((\PageIndex{10})\):

Solve by completing the square: \((x^2 + 3x + 4 = 0)\).

**Solution**

\[
\begin{aligned}
&x^2 + 3x + 4 = 0 \quad \text{The leading coefficient is 1} \quad \\
&x^2 + 3x = -4 \quad \text{Isolate the variable terms} \quad \\
&x^2 + 3x + \frac{9}{4} = -4 + \frac{9}{4} \quad \text{Add a constant - the square of half the coefficient of } x, \left(\frac{3}{2}\right)^2 = \frac{9}{4} \quad \\
&\left(x + \frac{3}{2}\right)^2 = \frac{-7}{4} \quad \text{Solve using the square root property} \quad \\
&x + \frac{3}{2} = \pm \frac{i \sqrt{7}}{2} \quad \text{Solution Set: } \left\{ -\frac{3}{2} \pm \frac{\sqrt{7}}{2} i \right\}
\end{aligned}
\]

So far, all of the examples have had a leading coefficient of \((1)\). If this is not the case, remove it. This can be done by dividing both sides of the equal sign by the leading coefficient before completing the square.

Example \((\PageIndex{11})\):

Solve by completing the square: \((3x^2 - 12x + 17 = 0)\).

**Solution**

\[
\begin{aligned}
3x^2 - 12x + 17 &= 0 \quad \text{The leading coefficient is NOT 1} \quad \\
\frac{3x^2 - 12x + 17}{3} &= \frac{0}{3} \quad \text{Divide by the coefficient to remove it from the } x^2 \text{ term} \quad \\
\frac{3x^2}{3} - 4x + \frac{17}{3} &= 0 \quad \\
x^2 - 4x &= -\frac{17}{3} \quad \text{Isolate the variable terms} \quad \\
\end{aligned}
\]
\[ x^2 - 4x + 4 = \frac{-17}{3} + 4 \]

\[
(x-2)(x-2) = \frac{-17}{3} + \frac{12}{3}
\]

\[
(x-2)^2 = \frac{-5}{3}
\]

\[
x = 2 \pm \frac{i\sqrt{15}}{\sqrt{3}}
\]

**Solution Set:** \( \{ \frac{6 \pm i\sqrt{15}}{3} \} \)

Example (PageIndex{12}):

Solve by completing the square: \(2x^2 + 5x - 1 = 0\).

**Solution**

\[
\begin{aligned}
2x^2 + 5x - 1 &= 0 &\text{The leading coefficient is NOT 1} \\ 
\frac{2x^2 + 5x - 1}{2} &= \frac{0}{2} &\text{Divide by the leading coefficient to remove it from the } x^2 \text{ term} \\
\frac{2x^2}{2} + \frac{5x}{2} - \frac{1}{2} &= 0 &\text{Isolate the variable terms} \\
x^2 + \frac{5}{2}x &= \frac{1}{2} &\text{Add a constant - the square of half the coefficient of } x, \\
x^2 + \frac{5}{2}x + \frac{25}{16} &= \frac{1}{2} + \frac{25}{16} &\text{Factor into a binomial square} \\
(x + \frac{5}{4})^2 &= \frac{33}{16} &\text{Solve using the square root property} \\
x + \frac{5}{4} &= \pm \sqrt{\frac{33}{16}} \\
x &= -\frac{5}{4} \pm \frac{\sqrt{33}}{4}
\end{aligned}
\]

**Solution Set:** \( \{ -\frac{5 \pm \sqrt{33}}{4} \} \)

Try It (PageIndex{13}):

Solve by completing the square.

a) \((x^2-6x=13)\)  
b) \(\left(x^2 - 2x - 17 = 0\right)\)  
c) \(\left(3x^2 + 2x - 1 = 0\right)\)
The Quadratic Formula

The fourth method of solving a **quadratic equation** is by using the **quadratic formula**, a formula that will solve all quadratic equations. Although the quadratic formula works on any quadratic equation in standard form, it is easy to make errors in substituting the values into the formula. Pay close attention when substituting, and use parentheses when inserting a negative number.

We can derive the quadratic formula by **completing the square**. Given \( ax^2+bx+c=0, \ a \neq 0 \), we will complete the square as follows:

\[
\begin{aligned}
ax^2+bx+c &= 0 & \text{The leading coefficient is NOT 1} \ \&
\left( x + \frac{b}{2a} \right)^2 &= \frac{b^2-4ac}{4a^2} & \text{Solve using the square root property} \\
\end{aligned}
\]

THE QUADRATIC FORMULA

Written in standard form, \( ax^2+bx+c=0 \) where \( (a), (b), \) and \( (c) \) are real numbers and \( (a\neq0) \), any quadratic equation can be solved using the **quadratic formula**:

\[ x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} \ nonumber \]

HowTo: Use the Quadratic Formula to solve an equation

1. Make sure the equation is in standard form: \( ax^2+bx+c=0 \).
2. Make note of the values of the coefficients and constant term, \( (a), (b), \) and \( (c) \).
3. Carefully substitute the values noted in step 2 into the equation. To avoid needless errors, use parentheses around...
each number input into the formula.

4. Calculate and solve.

Example (PageIndex{14}): Solve using the quadratic formula: \((2x^2-7x-15=0)\)

Solution

The coefficients are: \((a=2,b=-7,c=-15)\). Substitute these values into the quadratic formula and simplify.

\[
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
&= \frac{-(-7) \pm \sqrt{(-7)^2-4(2)(-15)}}{2(2)} \\
&= \frac{7 \pm \sqrt{169}}{4} \\
&= \frac{7 \pm 13}{4}
\end{aligned}
\]

\textbf{Solution Set: } \{-\frac{3}{2}, 5\}

\text{ because } x = \frac{7-13}{4} = \frac{-6}{4} = -\frac{3}{2} \text{ and } x = \frac{7+13}{4} = \frac{20}{4} = 5.

Example (PageIndex{15}): Solve using the quadratic formula: \((3x^2+6x-2=0)\).

Solution

The coefficients are: \((a=3, b=6, c=-2)\).

\[
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
&= \frac{-6 \pm \sqrt{60}}{6} \\
&= \frac{-6 \pm 2\sqrt{15}}{6} = \frac{-3 \pm \sqrt{15}}{3}
\end{aligned}
\]

Two ways the solution set can be written are: \(\Big\{\frac{-3 \pm \sqrt{15}}{3}\Big\}\) or \(\Big\{-1 \pm \frac{\sqrt{15}}{3}\Big\}\).

Sometimes terms are missing. When this is the case, use \(0\) as the coefficient. Also make sure the terms are written in descending powers of \(x\) so the correct values of the parameters \((a, b, c)\) are used in the formula.
Example $(\PageIndex{16})$:

Solve using the quadratic formula: $(45-x^2)=0$)

**Solution**

This equation is equivalent to $(-1 \cdot x^2 + 0 \cdot x + 45=0)$ so $(a=1 \quad b=0 \quad c=45)$

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
&= \frac{-(0) \pm \sqrt{(0)^2-4(-1)(45)}}{2(-1)} \\
&= \frac{0 \pm \sqrt{36(5)}}{-2} \\
&= \frac{\pm 6 \sqrt{5}}{2} \\
&= \pm 3 \sqrt{5} \\
\text{ Solution Set: } \{\pm 3 \sqrt{5}\}
\end{aligned}
$$

Often solutions to quadratic equations are not real.

Example $(\PageIndex{17})$:

Solve using the quadratic formula: $(x^2-4x+29)=0$).

**Solution**

The coefficients are: $(a=1 \quad b=-4 \quad c=29)$

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
&= \frac{-(-4) \pm \sqrt{(-4)^2-4(1)(29)}}{2(1)} \\
&= \frac{4 \pm \sqrt{-100}}{2} \\
&= \frac{4 \pm 10i}{2} \\
&= 2 \pm 5i \\
\text{ Solution Set: } \{2 \pm 5i\}
\end{aligned}
$$

Example $(\PageIndex{18})$:

Use the quadratic formula to solve $(x^2+x+2)=0$).

**Solution**

The coefficients are: $(a=1 \quad b=1 \quad c=2)$. Substitute these values into the quadratic formula.

$$
\begin{align*}
x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
&= \frac{-1 \pm \sqrt{1-8}}{2} \\
&= \frac{-1 \pm i\sqrt{7}}{2} \\
\text{ Solution Set: } \Big\{ \frac{-1 \pm i\sqrt{7}}{2} \Big\}
\end{align*}
$$
If multiple roots and complex roots are counted, then the fundamental theorem of algebra implies that every polynomial with one variable will have as many roots as its degree. For example, we expect $f(x) = x^3 - 8$ to have three roots. In other words, the equation

$$x^3 - 8 = 0$$

should have three solutions. To find them one might first think of trying to extract the cube roots just as we did with square roots,

$$x^3 - 8 = 0 \quad \Rightarrow \quad x^3 = 8 \quad \Rightarrow \quad x = \sqrt[3]{8} \quad \Rightarrow \quad x = 2$$

As you can see, this leads to one solution, the real cube root. There should be two others; let’s try to find them.

Example \(\PageIndex{19}\):

Find the set of all roots: $f(x) = x^3 - 8$.

Solution

Notice that the expression $x^3 - 8$ is a difference of cubes and recall that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. Here $a=x$ and $b=2$ and we can write

$$x^3 - 8 = 0 \quad \Rightarrow \quad (x-2)(x^2 + 2x + 4) = 0$$

Next apply the zero-product property and set each factor equal to zero. After setting the factors equal to zero we can then solve the resulting equation using the appropriate methods.

$$x - 2 = 0 \quad \text{or} \quad x^2 + 2x + 4 = 0$$

Solving the second equation using the quadratic formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{-12}}{2} = -1 \pm i\sqrt{3}$$

Using this method, we were able to obtain the set of all three roots $\{2, -1 \pm i\sqrt{3}\}$, one real and two complex.

Try It \(\PageIndex{20}\)

Solve the quadratic equation using the quadratic formula: $9x^2 + 3x - 2 = 0$.

Answer

$$x = -\frac{2}{3}, x = \frac{1}{3}$$

Sometimes factoring will produce a quadratic factor that needs to be solved using the quadratic formula or complete the square.
Example \(\PageIndex{21}\):

Solve \( 2x^4+2000x=0 \)

**Solution:**

Factor first. \( 2x^4+2000x = 2x(x^3+1000) = 2x(x+10)(x^2-10x+100) \)

Use the zero factor property and complete the square on: \( 2x(x+10)(x^2-10x+100) = 0 \)

\[
\begin{array}{lll}
2x=0 & (x+10) = 0 & (x^2-10x+100) = 0 \\
0 & x=-10 & x^2-10x+100 = -100+\square \\
&& (x-5)^2 = -100+25 \\
&& x-5 = \pm \sqrt{-75} \\
&& x = 5 \pm 5i\sqrt{3}
\end{array}
\]

The solution set is \( \{ 0, -10, 5 \pm 5i\sqrt{3} \} \)

---

**The Discriminant**

The **quadratic formula** not only generates the solutions to a quadratic equation, it tells us about the nature of the solutions when we consider the **discriminant**, or the expression under the radical, \( b^2−4ac \). The discriminant tells us whether the solutions are real numbers or complex numbers, and how many solutions of each type to expect. The table below relates the value of the discriminant to the solutions of a quadratic equation.

<table>
<thead>
<tr>
<th>Value of Discriminant</th>
<th>(b^2−4ac=0)</th>
<th>(b^2−4ac&gt;0), and is a perfect square</th>
<th>(b^2−4ac&gt;0), and is not a perfect square</th>
<th>(b^2−4ac&lt;0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Results</td>
<td>One rational solution (double solution)</td>
<td>Two rational solutions</td>
<td>Two irrational solutions</td>
<td>Two complex solutions</td>
</tr>
</tbody>
</table>

**THE DISCRIMINANT**

For \( ax^2+bx+c=0 \), where \( a \), \( b \), and \( c \) are real numbers, the **discriminant** is the expression under the radical in the quadratic formula: \( b^2−4ac \). It tells us whether the solutions are real numbers or complex numbers and how many solutions of each type to expect.

Example \(\PageIndex{20}\): Using the Discriminant to Find the Nature of the Solutions to a Quadratic Equation

Use the discriminant to find the nature of the solutions to the following quadratic equations:
Solution. Calculate the discriminant \(b^2-4ac\) for each equation and state the expected type of solutions.

a. \(x^2+4x+4=0\)
\(b^2-4ac=(4)^2-4(1)(4)=0\) There will be one rational double solution.

b. \(8x^2+14x+3=0\)
\(b^2-4ac=(14)^2-4(8)(3)=100\) As \(100\) is a perfect square, there will be two rational solutions.

c. \(3x^2-5x-2=0\)
\(b^2-4ac=(-5)^2-4(3)(-2)=49\) As \(49\) is a perfect square, there will be two rational solutions.

d. \(3x^2-10x+15=0\)
\(b^2-4ac=(-10)^2-4(3)(15)=-80\) There will be two complex solutions.