1.5: Equations with Rational Exponents

We have solved linear equations, rational equations, radical equations, and quadratic equations using several methods. However, there are many other types of equations, such as equations involving rational exponents, polynomial equations, absolute value equations, equations in quadratic form, and some rational equations that can be transformed into quadratics. Solving any equation, however, employs the same basic algebraic rules.

Solving Equations Involving Rational Exponents

Rational exponents are exponents that are fractions, where the numerator is a power and the denominator is a root. For example, \((16)^{\frac{1}{2}}\) is another way of writing \(\sqrt{16}\); \(8^{\frac{1}{3}}\) is another way of writing \(\sqrt[3]{8}\). The ability to work with rational exponents is a useful skill, as it is highly applicable in calculus.

Equations in which a variable expression is raised to a rational exponent can be solved by raising both sides of the equation to the reciprocal of the exponent. The reason the expression is raised to the reciprocal of its exponent is because the product of a number and its reciprocal is one. Therefore the exponent on the variable expression becomes one and is thus eliminated.

Definition: Rational Exponents

A **rational exponent** indicates a power in the numerator and a root in the denominator. There are multiple ways of writing an expression, a variable, or a number with a rational exponent:

\[
\left( a^{\frac{m}{n}} \right) = \left( a^{\frac{1}{n}} \right)^m = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m
\]
Example \(\PageIndex{1}\): Evaluate a Number Raised to a Rational Exponent

Evaluate \(8^{\frac{2}{3}}\)

**Solution.** It does not matter whether the root or the power is done first because \(8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = (8^{\frac{1}{3}})^2\). Since the cube root of \(8\) is easy to find, \(8^{\frac{2}{3}}\) can be evaluated as \((8^{\frac{1}{3}})^2 = (2)^2 = 4\).

Try It \(\PageIndex{1}\)

Evaluate \((64)^{\frac{-1}{3}}\)

Answer

\((\frac{1}{4})\)

**How to:** Solve an Equation with Rational Exponents.

1. **Isolate** the expression with the rational exponent
2. Raise both sides of the equation to the reciprocal power.
   - If the numerator of the reciprocal power is an even number, the solution must be checked because the solution involves the squaring process which can introduce extraneous roots.
   - If the denominator of the reciprocal power is an even number, this is equivalent to taking an even root so +/- must be included.

Example \(\PageIndex{2}\): Solve an Equation Containing a Variable Raised to a Rational Exponent

Solve the equation in which a variable is raised to a rational exponent: \(x^{\frac{3}{4}} = 8\).

**Solution** The exponent on \(x\) is removed by raising both sides of the equation to a power that is the reciprocal of \(\frac{3}{4}\). The reciprocal of \(\frac{3}{4}\) is \(\frac{4}{3}\). The numerator of this exponent we are applying is an even number, which means that both sides are being raised to an even power.

\[
\begin{align*}
\left(x^{\frac{3}{4}}\right)^{\frac{4}{3}} &= 8^{\frac{4}{3}} \\
x &= (8^{1/3})^4 \\
&= (2)^4 \\
&= 16
\end{align*}
\]

It is necessary to check our result because the solution involved raising both sides of the equation to an even power. Raising both sides of an equation to an even power can introduce "extraneous" roots. Therefore our answer must be checked: \((16^{\frac{3}{4}}) = (16^{\frac{1}{4}})^3 = 2^3 = 8\). \(\textcolor{Cerulean}{\text{?}}\) The solution set is \(\{16\}\).

Example \(\PageIndex{3}\)
Solve \((x^{\frac{5}{4}}+36=4)\).

**Solution**

\[
(x^{\frac{5}{4}}+36=4) \\
\Rightarrow (x^{\frac{5}{4}})^{\frac{4}{5}} =(-32)^{\frac{4}{5}} \\
\Rightarrow x = \left(\sqrt[5]{-32}\right)^4 \\
\Rightarrow x = (-2)^4 \\
\Rightarrow x = 16
\]

It is necessary in this case to check our result because the solution involved raising both sides of the equation to an even power. Raising both sides of an equation to an even power can introduce "extraneous" roots. \((16^{\frac{5}{4}}+36=\left(\sqrt[4]{16}\right)^5 +36= 2^5 +36 = 32 + 36 =68 \neq 4\). Therefore, the solution \(x=16\) must be rejected. Therefore this problem has no solution. The solution set is \(\{\quad \}\).

Example \(\PageIndex{4}\)

Solve \((x^{\frac{4}{3}}=81)\)

**Solution.** The solution involves raising both sides of the equal sign to the power of \(\frac{3}{4}\). Because the denominator is an even number, that means that we are actually taking the even root of a quantity, which could be either a positive or negative value.

\[
\Rightarrow (x^{\frac{4}{3}})=81^{\frac{3}{4}} \\
\Rightarrow x = \pm (3)^3 \\
\Rightarrow x = \pm 27
\]

No checking is required in this example because the process did not involve raising both sides of the equation to an even power. The even number was in the denominator, not the numerator of the reciprocal power. The solution set is \(\{ -27, 27 \}\).

Example \(\PageIndex{5}\)

Solve \(((x+5)^{\frac{2}{3}}= 64)\)

**Solution.** Notice here that the reciprocal power has an even denominator which represents taking the square root of both sides of the equation. This requires using \(\pm\) in the solution process.

\[
\Rightarrow ((x+5)^{\frac{2}{3}})=64^{\frac{2}{3}} \\
\Rightarrow x+5 =\pm (8)^3 \\
\Rightarrow x = -5+512 \text{ and } x = -5-512
\]
\[(x = 509) \text{ and } (x = -517)\]

The solution does not need to be checked! Solution Set: \(\{ 509, -517 \}\)

Try It

Solve the equation

a. \((x-4)^{\frac{2}{3}} = 25\)

b. \((x+5)^{\frac{3}{2}} = 8\)

c. \((x+12)^{\frac{3}{2}} = 8\)

Answer

a. \(\{ 129, -121 \}\)

b. \(\{ -1 \}\)

c. \(\{ \} \)

Example: Solve an Equation involving Rational Exponents and Factoring

Solve \(3x^{\frac{3}{4}} = x^{\frac{1}{2}}\).

Solution

This equation involves rational exponents as well as factoring rational exponents. Let us take this one step at a time. First, put the variable terms on one side of the equal sign and set the equation equal to zero.

\[
3x^{\frac{3}{4}} - x^{\frac{1}{2}} = 0
\]

Now, it looks like we should factor the left side, but what do we factor out? We can always factor the term with the lowest exponent. The factor with the lowest exponent is \(x^{1/2}\), so \(x^{3/4}\) needs to be rewritten as a product involving \(x^{1/2}\).

\[
3x^{\frac{1}{2}}x^{\frac{1}{4}} - x^{\frac{1}{2}} = 0
\]

Now we have two factors and can use the zero factor theorem.

\[
\begin{align*}
(x^{\frac{1}{2}})(3x^{\frac{1}{4}} - 1) &= 0 \\
(x^{\frac{1}{2}}) &= 0 \quad \text{or} \quad 3x^{\frac{1}{4}} - 1 = 0
\end{align*}
\]

Now we have two factors and can use the zero factor theorem.
& x&= \dfrac{1}{81} \\
\end{array} \\

The solution set is \( \{0, \dfrac{1}{81}\} \).