2.8: Theory of Existence and Uniqueness

Recall the theorem that says that if a first order differential satisfies continuity conditions, then the initial value problem will have a unique solution in some neighborhood of the initial value. More precisely,

Theorem: A Result For Nonlinear First Order Differential Equations

Let

\[ y' = f(x,y) \quad y(x_0) = y_0 \]

be a differential equation such that both partial derivatives

\[ f_x \quad \text{and} \quad f_y \]

are continuous in some rectangle containing \((x_0,y_0)\)/

Then there is a (possibly smaller) rectangle containing \((x_0,y_0)\) such that there is a unique solution \((f(x))\) that satisfies it.

Although a rigorous proof of this theorem is outside the scope of the class, we will show how to construct a solution to the initial value problem. First by translating the origin we can change the initial value problem to

\[ y(0) = 0. \]

Next we can change the question as follows. \((f(x))\) is a solution to the initial value problem if and only if

\[ f'(x) = f(x,f(x)) \quad \text{and} \quad f(0) = 0. \]
Now integrate both sides to get

\[ \phi(t) = \int_0^t f(s, \phi(s)) \, ds. \]

Notice that if such a function exists, then it satisfies \( \phi(0) = 0 \).

The equation above is called the **integral equation** associated with the differential equation.

It is easier to prove that the integral equation has a unique solution, then it is to show that the original differential equation has a unique solution. The strategy to find a solution is the following. First guess at a solution and call the first guess \( f_0(t) \). Then plug this solution into the integral to get a new function. If the new function is the same as the original guess, then we are done. Otherwise call the new function \( f_1(t) \). Next plug in \( f_1(t) \) into the integral to either get the same function or a new function \( f_2(t) \). Continue this process to get a sequence of functions \( f_n(t) \). Finally take the limit as \( n \) approaches infinity. This limit will be the solution to the integral equation. In symbols, define recursively

\[
\begin{align*}
  f_0(t) &= 0 \\
  \phi_{n+1}(t) &= \int_0^t f(s, \phi_n(s)) \, ds.
\end{align*}
\]

**Example**

Consider the differential equation

\[
y' = y + 2, \quad y(0) = 0.
\]

We write the corresponding integral equation

\[
y(t) = \int_0^t (y(s) + 2) \, ds.
\]

We choose

\[
f_0(t) = 0
\]

and calculate

\[
\begin{align*}
  \phi_1(t) &= \int_0^t (0 + 2) \, ds = 2t \\
  \phi_2(t) &= \int_0^t (2s + 2) \, ds = t^2 + 2t \\
  \phi_3(t) &= \int_0^t (s^2 + 2s + 2) \, ds = \frac{t^3}{3} + t^2 + 2t
\end{align*}
\]

and
\( \phi_4(t) = \int_0^t \left( \frac{s^3}{3} + s^2 + 2s + 2 \right) \, ds = \frac{t^4}{4.3.2} + \frac{t^3}{3.2} + \frac{t^2}{2} + \frac{t}{1} + \frac{1}{1} \)

Multiplying and dividing by 2 and adding 1 gives
\[
\frac{f_4(t)}{2} + 1 = \frac{t^4}{4.3.2} + \frac{t^3}{3.2} + \frac{t^2}{2} + \frac{t}{1} + \frac{1}{1}.
\]

The pattern indicates that
\[
\frac{f_n(t)}{2} + 1 = \sum \frac{t^n}{n!}
\]
or
\[
\frac{f(t)}{2} + 1 = e^t.
\]

Solving we get
\[
[f(t) = 2(e^t - 1)].
\]

This may seem like a proof of the uniqueness and existence theorem, but we need to be sure of several details for a true proof.

1. Does \( f_n(t) \) exist for all \( n \)? Although we know that \( f(t,y) \) is continuous near the initial value, the integral could possibly result in a value that lies outside this rectangle of continuity. This is why we may have to get a smaller rectangle.
2. Does the sequence \( f_n(t) \) converge? The limit may not exist.
3. If the sequence \( f_n(t) \) does converge, is the limit continuous?
4. Is \( f(t) \) the only solution to the integral equation?

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**Contributors**

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