2.8: Theory of Existence and Uniqueness

Recall the theorem that says that if a first order differential satisfies continuity conditions, then the initial value problem will have a unique solution in some neighborhood of the initial value. More precisely,

**Theorem: A Result For Nonlinear First Order Differential Equations**

Let

\[
\begin{align*}
y' &= f(x,y) \\
y(x_0) &= y_0
\end{align*}
\]

be a differential equation such that both partial derivatives

\[
\begin{align*}
f_x \\
f_y
\end{align*}
\]

are continuous in some rectangle containing \((x_0,y_0)\). Then there is a (possibly smaller) rectangle containing \((x_0,y_0)\) such that there is a unique solution \(f(x)\) that satisfies it.

Although a rigorous proof of this theorem is outside the scope of the class, we will show how to construct a solution to the initial value problem. First by translating the origin we can change the initial value problem to

\[
y(0) = 0
\]

Next we can change the question as follows. \(f(x)\) is a solution to the initial value problem if and only if

\[
f'(x) = f(x,f(x)) \quad \text{and} \quad f(0) = 0
\]
Now integrate both sides to get
\[ \phi(t) = \int_0^t f(s, \phi(s)) \, ds \, . \]

Notice that if such a function exists, then it satisfies \( \phi(0) = 0 \).

The equation above is called the integral equation associated with the differential equation.

It is easier to prove that the integral equation has a unique solution, then it is to show that the original differential equation has a unique solution. The strategy to find a solution is the following. First guess at a solution and call the first guess \( \phi_0(t) \). Then plug this solution into the integral to get a new function. If the new function is the same as the original guess, then we are done. Otherwise call the new function \( \phi_1(t) \). Next plug in \( \phi_1(t) \) into the integral to either get the same function or a new function \( \phi_2(t) \). Continue this process to get a sequence of functions \( \{ \phi_n(t) \} \). Finally take the limit as \( n \) approaches infinity. This limit will be the solution to the integral equation. In symbols, define recursively

\[ \phi_0(t) = 0 \, . \]

\[ \phi_{n+1}(t) = \int_0^t f(s, \phi_n(s)) \, ds \, . \]

Example (PageIndex{1})

Consider the differential equation
\[ y' = y + 2, \quad y(0) = 0 \, . \]

We write the corresponding integral equation
\[ y(t) = \int_0^t (y(s) + 2) \, ds \, . \]

We choose
\[ \phi_0(t) = 0 \, . \]

and calculate
\[ \phi_1(t) = \int_0^t 2 \, ds = 2t \, . \]

and
\[ \phi_2(t) = \int_0^t (2s + 2) \, ds = t^2 + 2t \, . \]

and
\[ \phi_3(t) = \int_0^t (s^2 + 2s + 2) \, ds = \frac{t^3}{3} + t^2 + 2t \, . \]

and

\[ \phi_4(t) = \int_0^t \left(\frac{s^3}{3}+s^2+2s+2 \right) \, ds = \frac{t^4}{3} + \frac{t^3}{3} + t^2 + 2t. \]

Multiplying and dividing by 2 and adding 1 gives
\[ \frac{f_4(t)}{2} + 1 = \frac{t^4}{4.3.2} + \frac{t^3}{3.2} + \frac{t^2}{2} + \frac{t}{1} + \frac{1}{1}. \]

The pattern indicates that
\[ \frac{f_n(t)}{2} + 1 = \sum \frac{t^n}{n!} \]
or
\[ \frac{f(t)}{2} + 1 = e^t. \]

Solving we get
\[ f(t) = 2(e^t - 1). \]

This may seem like a proof of the uniqueness and existence theorem, but we need to be sure of several details for a true proof.

1. Does \( f_n(t) \) exist for all \( n \)? Although we know that \( f(t,y) \) is continuous near the initial value, the integral could possibly result in a value that lies outside this rectangle of continuity. This is why we may have to get a smaller rectangle.
2. Does the sequence \( f_n(t) \) converge? The limit may not exist.
3. If the sequence \( f_n(t) \) does converge, is the limit continuous?
4. Is \( f(t) \) the only solution to the integral equation?

Contributors

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