2.8: Theory of Existence and Uniqueness

Recall the theorem that says that if a first order differential satisfies continuity conditions, then the initial value problem will have a unique solution in some neighborhood of the initial value. More precisely,

Theorem: A Result For Nonlinear First Order Differential Equations

Let

\[ y' = f(x,y) \quad \text{and} \quad y(x_0) = y_0 \]

be a differential equation such that both partial derivatives

\[ f_x \quad \text{and} \quad f_y \]

are continuous in some rectangle containing \((x_0,y_0)\).

Then there is a (possibly smaller) rectangle containing \((x_0,y_0)\) such that there is a unique solution \(f(x)\) that satisfies it.

Although a rigorous proof of this theorem is outside the scope of the class, we will show how to construct a solution to the initial value problem. First by translating the origin we can change the initial value problem to

\[ y(0) = 0. \]

Next we can change the question as follows. \(f(x)\) is a solution to the initial value problem if and only if

\[ f'(x) = f(x,f(x)) \quad \text{and} \quad f(0) = 0. \]
Now integrate both sides to get

\[
\phi (t) = \int _0^t f(s,\phi (s)) \, ds .
\]

Notice that if such a function exists, then it satisfies \( \phi(0) = 0 \).

The equation above is called the **integral equation** associated with the differential equation.

It is easier to prove that the integral equation has a unique solution, then it is to show that the original differential equation has a unique solution. The strategy to find a solution is the following. First guess at a solution and call the first guess \( f_0(t) \). Then plug this solution into the integral to get a new function. If the new function is the same as the original guess, then we are done. Otherwise call the new function \( f_1(t) \). Next plug in \( f_1(t) \) into the integral to either get the same function or a new function \( f_2(t) \). Continue this process to get a sequence of functions \( \{ f_n(t) \} \). Finally take the limit as \( n \) approaches infinity. This limit will be the solution to the integral equation. In symbols, define recursively

\[
f_0(t) = 0
\]

\[
\phi_{n+1} (t) = \int _0^t f(s,\phi_n (s)) \, ds .
\]

**Example**

Consider the differential equation

\[y' = y + 2, \quad y(0) = 0.\]

We write the corresponding integral equation

\[y(t) = \int_0^t (y(s)+2) \, ds .\]

We choose

\[f_0(t) = 0\]

and calculate

\[\phi_1(t) = \int_0^t (0+2) \, ds = 2t\]

and

\[\phi_2(t) = \int_0^t (2s+2) \, ds = t^2 + 2t\]

and

\[\phi_3(t) = \int_0^t (s^2+2s+2) \, ds = \frac{t^3}{3}+t^2 + 2t\]

and
\[ \phi_4(t) = \int_0^t \left(\frac{s^3}{3} + s^2 + 2s + 2 \right) \, ds = \frac{t^4}{4.3.2} + \frac{t^3}{3.2} + \frac{t^2}{2} + \frac{t}{1} + \frac{1}{1}. \]

Multiplying and dividing by 2 and adding 1 gives

\[ \frac{f_4(t)}{2} + 1 = \frac{t^4}{4.3.2} + \frac{t^3}{3.2} + \frac{t^2}{2} + \frac{t}{1} + \frac{1}{1}. \]

The pattern indicates that

\[ \frac{f_n(t)}{2} + 1 = \sum \frac{t^n}{n!} \]

or

\[ \frac{f(t)}{2} + 1 = e^t. \]

Solving we get

\[ f(t) = 2(e^t - 1). \]

This may seem like a proof of the uniqueness and existence theorem, but we need to be sure of several details for a true proof.

1. Does \( f_n(t) \) exist for all \( n \)? Although we know that \( f(t, y) \) is continuous near the initial value, the integral could possibly result in a value that lies outside this rectangle of continuity. This is why we may have to get a smaller rectangle.

2. Does the sequence \( f_n(t) \) converge? The limit may not exist.

3. If the sequence \( f_n(t) \) does converge, is the limit continuous?

4. Is \( f(t) \) the only solution to the integral equation?

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