2.8: Theory of Existence and Uniqueness

Recall the theorem that says that if a first order differential satisfies continuity conditions, then the initial value problem will have a unique solution in some neighborhood of the initial value. More precisely,

**Theorem: A Result For Nonlinear First Order Differential Equations**

Let

\[ y' = f(x, y) \quad ; \quad y(x_0) = y_0 \]

be a differential equation such that both partial derivatives

\[ f_x \quad \text{and} \quad f_y \]

are continuous in some rectangle containing \((x_0, y_0)/\)

Then there is a (possibly smaller) rectangle containing \((x_0, y_0)/\) such that there is a unique solution \((f(x))/\) that satisfies it.

Although a rigorous proof of this theorem is outside the scope of the class, we will show how to construct a solution to the initial value problem. First by translating the origin we can change the initial value problem to

\[ y(0) = 0. \]

Next we can change the question as follows. \((f(x))/\) is a solution to the initial value problem if and only if

\[ f'(x) = f(x, f(x)) \quad \text{and} \quad f(0) = 0. \]
Now integrate both sides to get
\[
\phi(t) = \int_0^t f(s, \phi(s)) \, ds.
\]

Notice that if such a function exists, then it satisfies \(\phi(0) = 0\).

The equation above is called the integral equation associated with the differential equation.

It is easier to prove that the integral equation has a unique solution, then it is to show that the original differential equation has a unique solution. The strategy to find a solution is the following. First guess at a solution and call the first guess \(f_0(t)\). Then plug this solution into the integral to get a new function. If the new function is the same as the original guess, then we are done. Otherwise call the new function \(f_1(t)\). Next plug in \(f_1(t)\) into the integral to either get the same function or a new function \(f_2(t)\). Continue this process to get a sequence of functions \(f_n(t)\). Finally take the limit as \(n\) approaches infinity. This limit will be the solution to the integral equation. In symbols, define recursively

\[
f_0(t) = 0
\]

\[
\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s)) \, ds.
\]

Example (PageIndex{1})

Consider the differential equation
\[
y' = y + 2, \quad y(0) = 0.
\]

We write the corresponding integral equation
\[
y(t) = \int_0^t (y(s) + 2) \, ds.
\]

We choose
\[
f_0(t) = 0
\]

and calculate
\[
\phi_1(t) = \int_0^t (0 + 2) \, ds = 2t
\]

and
\[
\phi_2(t) = \int_0^t (2s + 2) \, ds = t^2 + 2t
\]

and
\[
\phi_3(t) = \int_0^t (s^2 + 2s + 2) \, ds = \frac{t^3}{3} + t^2 + 2t
\]

and
\[ \phi_4(t) = \int_0^t \left( \frac{s^3}{3} + s^2 + 2s + 2 \right) \, ds = \frac{t^4}{3} + \frac{t^3}{3} + t^2 + 2t. \]

Multiplying and dividing by 2 and adding 1 gives

\[ \frac{f_4(t)}{2} + 1 = \frac{t^4}{4.3.2} + \frac{t^3}{3.2} + \frac{t^2}{2} + \frac{t}{1} + \frac{1}{1}. \]

The pattern indicates that

\[ \frac{f_n(t)}{2} + 1 = \sum \frac{t^n}{n!} \]

or

\[ \frac{f(t)}{2} + 1 = e^t. \]

Solving we get

\[ f(t) = 2(e^t - 1). \]

This may seem like a proof of the uniqueness and existence theorem, but we need to be sure of several details for a true proof.

1. Does \( f_n(t) \) exist for all \( n \)? Although we know that \( f(t,y) \) is continuous near the initial value, the integral could possibly result in a value that lies outside this rectangle of continuity. This is why we may have to get a smaller rectangle.
2. Does the sequence \( f_n(t) \) converge? The limit may not exist.
3. If the sequence \( f_n(t) \) does converge, is the limit continuous?
4. Is \( f(t) \) the only solution to the integral equation?

**Contributors**

- Larry Green (Lake Tahoe Community College)