3.2: Complex Roots of the Characteristic Equation

We have already addressed how to solve a second order linear homogeneous differential equation with constant coefficients where the roots of the characteristic equation are real and distinct. We will now explain how to handle these differential equations when the roots are complex. The example below demonstrates the method.

Example \( \PageIndex{1} \)

Solve

\[
 y'' - 4y' + 13y = 0.
\]

Solution

As before we assume that \( y = e^{rt} \) is a solution. We have

\[
 y' = re^{rt}; \quad y'' = r^2e^{rt}.
\]

Substituting back into the original differential equation gives

\[
 r^2e^{rt} - 4re^{rt} + 13e^{rt} = 0 \quad \text{dividing by } e^{rt}.
\]

This quadratic does not factor, so we use the quadratic formula and get the roots

\[
 r = 2 + 3i; \quad r = 2 - 3i.
\]
We can conclude that the general solution to the differential equation is

\[
\begin{align*}
y &= a_1e^{(2 + 3i)t} + a_2e^{(2-3i)t} \\
&= e^{2t} (a_1e^{3it} + a_2e^{-3it}) & \text{Factoring out the } e^{2t}. 
\end{align*}
\]

Although this gives the general solution, it is not satisfactory since the solution involves complex exponents. To deal with this we use Euler’s formula

\[e^{iq} = \cos q + i \sin q, q. \nonumber\]

This gives

\[y = e^{2t} \left[ a_1(\cos (3t) + i \sin (3t)) + a_2 (\cos (-3t) + i \sin (-3t)) \right]. \nonumber\]

Since the \( \cos x \) is an even function and \( \sin x \) is an odd function, we get

\[y = e^{2t} \left[ a_1(\cos (3t) + i \sin (3t)) + a_2 (\cos (3t) - i \sin (3t)) \right] \nonumber\]

or

\[y = e^{2t} \left[ (a_1 + a_2) \cos (3t) + (a_1 - a_2)i \sin (3t) \right]. \nonumber\]

Finally let

\[c_1 = a_1 + a_2 \; ; \; c_2 = i(a_1 - a_2) \nonumber\]

and we get

\[y = e^{2t} \left[ c_1 \cos (3t) + c_2 \sin (3t) \right]. \nonumber\]

General Solution

In general if

\[ay'' + by' + cy = 0 \]

is a second order linear differential equation with constant coefficients such that the characteristic equation has complex roots

\[r = l + mi \; ; \; r = l - mi \]

Then the general solution to the differential equation is given by

\[y = e^{lt}\left[ c_1 \cos(mt) + c_2 \sin(mt) \right]. \nonumber\]

Example \(\PageIndex{2}\): Graphical Solutions
Solve

\[ y'' - 10y' + 29 = 0 \text{ given } y(0) = 1, \quad y'(0) = 3 \]

**Solution**

The characteristic equation is

\[ r^2 - 10r + 29 = 0 \]

which has roots

\[ r = 5 + 2i \quad \text{and} \quad r = 5 - 2i. \]

The general solution is

\[ y = e^{5t}[ c_1 \cos (2t) + c_2 \sin (2t) ]. \]

We use the initial values to find the constants. Plug in \( y(0) = 1 \)

\[ 1 = 1 [ c_1 (1) + c_2 (0) ] \]

so that \( c_1 = 1 \). We have

\[ y' = 5e^{5t}[ \cos (2t) + c_2 \sin (2t) ] + e^{5t}[ -2 \sin (2t) + 2c_2 \cos (2t)]. \]

Plugging in \( y'(0) = 3 \)

\[ 3 = 5 [ 1 + 0] + 1[0 + 2c_2] \]

Hence \( c_2 = -1 \).

The final solution is

\[ y = e^{5t} [ \cos (2t) - \sin (2t) ] \]

Let's investigate the graphs of solutions for the general solutions in Equation \ref{gen1}

- For \( l = 0 \), the graph is periodic.
- For \( l > 0 \), the amplitude increases exponentially
- For \( l < 0 \), the amplitude approaches 0 exponentially.

The three types are pictured below.
Contributors and Attributions

- Larry Green (Lake Tahoe Community College)