3.1: Homogeneous Equations with Constant Coefficients

Up until now, we have only worked on first order differential equations. The next step is to investigate second order differential equations. The general second order differential equation has the form

\[ y'' = f(t,y,y') \label{1} \]

The general solution to such an equation is very difficult to identify. Instead, we will focus on special cases. In particular, if the differential equation is linear, then it can be written in the form

\[ P(t)y'' + Q(t)y' + R(t)y = G(t) \label{2} \]

If \( P(t) \) is nonzero, then we can divide by \( P(t) \) to get

\[ y'' + p(t)y' + q(t)y = g(t) \]

We call a second order linear differential equation homogeneous if \( g(t) = 0 \). In this section we will be investigating homogeneous second order linear differential equations with constant coefficients, which can be written in the form:

\[ ay'' + by' + cy = 0 \]

Example: General Solution

Solve

\[ y'' + 3y' - 4y = 0 \]
**Solution**

The strategy is to search for a solution of the form

\[ y = e^{rt} \]

The reason for this is that long ago some geniuses figured this stuff out and it works.

Now calculate derivatives

\[ y' = re^{rt} \]
\[ y'' = r^2e^{rt} \]

Substituting into the differential equation gives

\[ r^2e^{rt} + 3(re^{rt}) - 4(e^{rt}) = ( r^2 + 3r - 4)e^{rt} = 0 \]

Now divide by \(e^{rt}\) to get

\[ r^2 + 3r - 4 = 0 \]
\[ (r - 1)(r + 4) = 0 \]

The solutions (roots) to this polynomial are

\[ r = 1 \]
\[ r = -4 \]

We can conclude that two solutions are

\[ y_1 = e^t \]
\[ y_2 = e^{-4t} \]

Now let

\[ L(y) = y'' + 3y' - 4 \]

It is easy to verify that if \( y_1 \) and \( y_2 \) are solutions to
\[ L(y) = 0 \]

then

\[ c_1y_1 + c_2y_2 \]

is also a solution. More specifically we can conclude that

\[ y = c_1e^t + c_2e^{-4t} \]

Represents a two dimensional family (vector space) of solutions. Later we will prove that this is the most general description of the solution space.

Example \(\PageIndex{2}\): Initial Conditions

Solve

\[ y'' - y' - 6y = 0 \]

with \(y(0) = 1\) and \(y'(0) = 2\).

**Solution**

As before we seek solutions of the form

\[ y = e^{rt} \]

Now calculate derivatives

\[ y' = re^{rt} \]
\[ y'' = r^2e^{rt} \]

Substituting into the differential equation gives

\[ r^2e^{rt} + (re^{rt}) - 6(e^{rt}) = 0 \]

Now divide by \(e^{rt}\) to get

\[ (r^2 - r - 6)e^{rt} = 0 \]

We can conclude that two solutions are

\[ y_1 = e^{3t} \]

\[ y_2 = e^{-2t} \]
and

\[ y_2 = e^{-2t} \]

We can conclude that

\[ y = c_1e^{3t} + C_2e^{-2t} \]

Represents a two dimensional family (vector space) of solutions. Now use the initial conditions to find that

\[ 1 = c_1 + c_2 \]

We have that

\[ y' = 3C_1e^{3t} - 2C_2e^{-2t} \]

Plugging in the initial condition with \( y' \), gives

\[ 2 = 3c_1 - 2c_2 \]

This is a system of two equations and two unknowns. We can use a matrix to arrive at \( c_1 = \frac{4}{5} \) and \( C_2 = \frac{1}{5} \)

The final solution is

\[ y = \frac{4}{5} e^{3t} + \frac{1}{5}e^{-2t} \]

In general for

\[ ay'' + by' + cy = 0 \]

we call

\[ ar^2 + br + c = 0 \]

the characteristic equation for this differential equation. Our examples demonstrated how to solve it if we have two distinct real roots. For complex or repeated roots, a somewhat different strategy is needed; we will discuss these other cases later on. For real distinct roots we can use the quadratic formula an obtain a general solution

\[ y = c_1e^{r_1t} + c_2e^{r_2t} \]

where

\[ r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]

and
\[ r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]