3.6: Linear Independence and the Wronskian

Recall from linear algebra that two vectors \( \mathbf{v} \) and \( \mathbf{w} \) are called linearly dependent if there are nonzero constants \( c_1 \) and \( c_2 \) with

\[
\begin{align*}
  c_1 \mathbf{v} + c_2 \mathbf{w} &= 0.
\end{align*}
\]

We can think of differentiable functions \( f(t) \) and \( g(t) \) as being vectors in the vector space of differentiable functions. The analogous definition is below.

Definition: Linear Dependence and Independence

Let \( f(t) \) and \( g(t) \) be differentiable functions. Then they are called **linearly dependent** if there are nonzero constants \( c_1 \) and \( c_2 \) with \( c_1 f(t) + c_2 g(t) = 0 \) for all \( t \). Otherwise they are called **linearly independent**.

Example

The functions \( f(t) = 2\sin^2 t \) and \( g(t) = 1 - \cos^2(t) \) are linearly dependent since

\[
\begin{align*}
  (1)(2\sin^2 t) + (-2)(1 - \cos^2(t)) &= 0.
\end{align*}
\]

Example

The functions \( f(t) = t \) and \( g(t) = t^2 \) are linearly independent since otherwise there would be nonzero constants \( c_1 \) and \( c_2 \) such that

\[
\begin{align*}
  c_1 t + c_2 t^2 &= 0.
\end{align*}
\]
for all values of \(t\). First let \(t = 1\). Then
\[
\begin{align*}
  c_1 + c_2 &= 0. \\
\end{align*}
\]
Now let \(t = 2\). Then
\[
\begin{align*}
  2c_1 + 4c_2 &= 0. \\
\end{align*}
\]
This is a system of 2 equations and two unknowns. The determinant of the corresponding matrix is
\[
\begin{align*}
  4 - 2 &= 2. \\
\end{align*}
\]
Since the determinant is nonzero, the only solution is the trivial solution. That is
\[
\begin{align*}
  c_1 = c_2 = 0. \\
\end{align*}
\]
The two functions are linearly independent.

In the above example, we arbitarily selected two values for \(t\). It turns out that there is a systematic way to check for linear dependence. The following theorem states this way.

**Theorem**

Let \(f\) and \(g\) be differentiable on \([a,b]\). If Wronskian \(W(f,g)(t_0)\) is nonzero for some \(t_0\) in \([a,b]\) then \(f\) and \(g\) are linearly independent on \([a,b]\). If \(f\) and \(g\) are linearly dependent then the Wronskian is zero for all \(t\) in \([a,b]\).

**Example \(\PageIndex{3}\)**

Show that the functions \(f(t) = t\) and \(g(t) = e^{2t}\) are linearly independent.

**Solution**

We compute the Wronskian.
\[
\begin{align*}
  f'(t) &= 1 \\
  g'(t) &= 2e^{2t} \\
\end{align*}
\]

The Wronskian is
\[
\begin{align*}
  W(f,g)(t) &= (t)(2e^{2t}) - (e^{2t})(1) \\
\end{align*}
\]
Now plug in \(t=0\) to get
\[
\begin{align*}
  W(f,g)(0) &= -1 \\
\end{align*}
\]
which is nonzero. We can conclude that \(f\) and \(g\) are linearly independent.
Proof

If

\[ C_1 f(t) + C_2 g(t) = 0 \]

Then we can take derivatives of both sides to get

\[ C_1 f''(t) + C_2 g'(t) = 0 \]

This is a system of two equations with two unknowns. The determinant of the corresponding matrix is the Wronskian. Hence, if the Wronskian is nonzero at some \( t_0 \), only the trivial solution exists. Hence they are linearly independent.

\( \square \)

There is a fascinating relationship between second order linear differential equations and the Wronskian. This relationship is stated below.

Theorem: Abel's Theorem

Let \( (y_1) \) and \( (y_2) \) be solutions on the differential equation

\[ L(y) = y'' + p(t)y' + q(t)y = 0 \]

where \( (p) \) and \( (q) \) are continuous on \( (a,b) \). Then the Wronskian is given by

\[ W(y_1, y_2)(t) = ce^{-\int p(t) dt} \]

where \( (c) \) is a constant depending on only \( (y_1) \) and \( (y_2) \), but not on \( (t) \). The Wronskian is either zero for all \( (t) \) in \( (a,b) \) or not in \( (a,b) \).

Proof

First the Wronskian

\[ W = y_1y_2' - y_1'y_2 \]

has derivative

\[ W' = y_1y_2'' + y_1'y_2' - y_1'y_2' - y_1y_2'' = y_1'y_2' - y_1y_2''. \]

Since \( (y_1) \) and \( (y_2) \) are solutions to the differential equation, we have

\[ y''_1 + p(t)y'_1 + q(t)y_1 = 0 \]

\[ y''_2 + p(t)y'_2 + q(t)y_2 = 0 \]
Multiplying the first equation by \((-y_2)\) and the second by \((y_1)\) and adding gives

\[
( y_1y''_2 - y''_1y_2) + p(t)(y_1y'_2 - y_1y_2) = 0.
\]

This can be written as

\[
W' + p(t)W = 0.
\]

This is a separable differential equation with

\[
dfrac{dW}{W} = -p(t) \, dt.
\]

Now integrate and Abel's theorem appears.

Example \(\PageIndex{4}\)

Find the Wronskian (up to a constant) of the differential equations

\[
y'' + \cos(t) \, y = 0.
\]

Solution

We just use Abel's theorem, the integral of \(\cos(t)\) is \(\sin(t)\) hence the Wronskian is

\[
W(t) = ce^{\sin(t)}.
\]

A corollary of Abel's theorem is the following

Corollary

Let \((y_1)\) and \((y_2)\) be solutions to the differential equation

\[
L(y) = y'' + p(t)y' + q(t)y = 0
\]

Then either \((W(y_1, y_2))\) is zero for all \((t)\) or never zero.

Example \(\PageIndex{5}\)

Prove that

\[
y_1(t) = 1 - t \; \text{and} \; y_2(t) = t^3
\]

cannot both be solutions to a differential equation.
\[ y'' + p(t)y + q(t) = 0 \]

for \( p(t) \) and \( q(t) \) continuous on \( \left[ -1, 5 \right] \).

**Solution**

We compute the Wronskian

\[ y'_1 = -1 \quad \text{and} \quad y'_2 = 3t^2 \]

\[ W(y_1, y_2) = (1 - t)(3t^2) - (t^3)(-1) = 3t^2 - 2t^3. \]

Notice that the Wronskian is zero at \( t = 0 \) but nonzero at \( t = 1 \). By the above corollary, \( y_1 \) and \( y_2 \) cannot both be solutions.

**Contributors**

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