3.6: Linear Independence and the Wronskian

Recall from linear algebra that two vectors \(v\) and \(w\) are called linearly dependent if there are nonzero constants \(c_1\) and \(c_2\) with
\[
\begin{align*}
\begin{bmatrix}
c_1v + c_2w = 0.
\end{bmatrix}
\end{align*}
\]

We can think of differentiable functions \(f(t)\) and \(g(t)\) as being vectors in the vector space of differentiable functions. The analogous definition is below.

Definition: Linear Dependence and Independence

Let \(f(t)\) and \(g(t)\) be differentiable functions. Then they are called linearly dependent if there are nonzero constants \(c_1\) and \(c_2\) with \(c_1f(t) + c_2g(t) = 0\) for all \(t\). Otherwise they are called linearly independent.

Example \(\PageIndex{1}\)

The functions \(f(t) = 2\sin^2 t\) and \(g(t) = 1 - \cos^2(t)\) are linearly dependent since
\[
\begin{align*}
\begin{bmatrix}
(1)(2\sin^2 t) + (-2)(1 - \cos^2(t)) = 0.
\end{bmatrix}
\end{align*}
\]

Example \(\PageIndex{1}\)

The functions \(f(t) = t\) and \(g(t) = t^2\) are linearly independent since otherwise there would be nonzero constants \(c_1\) and \(c_2\) such that
\[
\begin{align*}
\begin{bmatrix}
c_1t + c_2t^2 = 0
\end{bmatrix}
\end{align*}
\]
for all values of \((t)\). First let \((t = 1)\). Then

\[
[ c_1 + c_2 = 0 \nonumber ]
\]

Now let \((t = 2)\). Then

\[
[ 2c_1 + 4c_2 = 0 \nonumber ]
\]

This is a system of 2 equations and two unknowns. The determinant of the corresponding matrix is

\[
[4 - 2 = 2 \nonumber ]
\]

Since the determinant is nonzero, the only solution is the trivial solution. That is

\[
[ c_1 = c_2 = 0 \nonumber ]
\]

The two functions are linearly independent.

In the above example, we arbitrarily selected two values for \((t)\). It turns out that there is a systematic way to check for linear dependence. The following theorem states this way.

**Theorem**

Let \(f\) and \(g\) be differentiable on \([a,b]\). If Wronskian \((W(f,g)(t_0))\) is nonzero for some \((t_0)\) in \([a,b]\) then \(f\) and \(g\) are linearly independent on \([a,b]\). If \(f\) and \(g\) are linearly dependent then the Wronskian is zero for all \((t)\) in \([a,b]\).

**Example**

Show that the functions \(f(t) = t\) and \(g(t) = e^{2t}\) are linearly independent.

**Solution**

We compute the Wronskian.

\[
[ f(t) = 1 g'(t) = 2e^{2t} \nonumber ]
\]

The Wronskian is

\[
[ (t)(2e^{2t}) - (e^{2t})(1) \nonumber ]
\]

Now plug in \((t=0)\) to get

\[
[ W(f, g)(0) = -1 \nonumber ]
\]

which is nonzero. We can conclude that \((f)\) and \((g)\) are *linearly independent*. 
Proof

If

\[ C_1 f(t) + C_2 g(t) = 0 \]

Then we can take derivatives of both sides to get

\[ C_1 f''(t) + C_2 g'(t) = 0 \]

This is a system of two equations with two unknowns. The determinant of the corresponding matrix is the Wronskian. Hence, if the Wronskian is nonzero at some \( t_0 \), only the trivial solution exists. Hence they are linearly independent.

\( \square \)

There is a fascinating relationship between second order linear differential equations and the Wronskian. This relationship is stated below.

Theorem: Abel's Theorem

Let \( y_1 \) and \( y_2 \) be solutions on the differential equation

\[ L(y) = y'' + p(t)y' + q(t)y = 0 \]

where \( p(t) \) and \( q(t) \) are continuous on \([a,b]\). Then the Wronskian is given by

\[ W_{y_1, y_2}(t) = c e^{-\int p(t) dt} \]

where \( c \) is a constant depending on only \( y_1 \) and \( y_2 \), but not on \( t \). The Wronskian is either zero for all \( t \) in \([a,b]\) or not in \([a,b]\).

Proof

First the Wronskian

\[ W = y_1 y_2' - y_1' y_2 \]

has derivative

\[ W' = y_1 y_2'' + y_1'' y_2 - y_1' y_2' - y_1 y_2'' = y_1 y_2'' - y_1'' y_2 \]

Since \( y_1 \) and \( y_2 \) are solutions to the differential equation, we have

\[ y_1'' + p(t)y_1' + q(t)y_1 = 0 \]
\[ y_2'' + p(t)y_2' + q(t)y_2 = 0 \]
Multiplying the first equation by \(-y_2\) and the second by \(y_1\) and adding gives

\[ (y_1y''_2 - y''_1y_2) + p(t)(y_1y'_2 - y_1y_2) = 0. \]

This can be written as

\[ W' + p(t)W = 0. \]

This is a separable differential equation with

\[ \frac{dW}{W} = -p(t) \, dt. \]

Now integrate and Abel's theorem appears.

Example (PageIndex{4})

Find the Wronskian (up to a constant) of the differential equations

\[ y'' + \cos(t) \, y = 0. \]

Solution

We just use Abel's theorem, the integral of \(\cos t\) is \(\sin t\) hence the Wronskian is

\[ W(t) = ce^{\sin t}. \]

A corollary of Abel's theorem is the following

Corollary

Let \((y_1)\) and \((y_2)\) be solutions to the differential equation

\[ L(y) = y'' + p(t)y' + q(t)y = 0 \]

Then either \(W(y_1, y_2)\) is zero for all \(t\) or never zero.

Example (PageIndex{5})

Prove that

\[ y_1(t) = 1 - t \] \( \text{and} \) \( y_2(t) = t^3 \)

cannot both be solutions to a differential equation
\[ y'' + p(t)y + q(t) = 0 \]

for \( p(t) \) and \( q(t) \) continuous on \( [-1, 5] \).

**Solution**

We compute the Wronskian

\[ y'_1 = -1 \quad \text{and} \quad y'_2 = 3t^2 \]

\[ W(y_1, y_2) = (1 - t)(3t^2) - (t^3)(-1) = 3t^2 - 2t^3. \]

Notice that the Wronskian is zero at \( t = 0 \) but nonzero at \( t = 1 \). By the above corollary, \( y_1 \) and \( y_2 \) cannot both be solutions.

**Contributors**

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