3.6: Linear Independence and the Wronskian

Recall from linear algebra that two vectors $\mathbf{v}$ and $\mathbf{w}$ are called linearly dependent if there are nonzero constants $c_1$ and $c_2$ with

$$c_1\mathbf{v} + c_2\mathbf{w} = 0.$$

We can think of differentiable functions $f(t)$ and $g(t)$ as being vectors in the vector space of differentiable functions. The analogous definition is below.

Definition: Linear Dependence and Independence

Let $f(t)$ and $g(t)$ be differentiable functions. Then they are called **linearly dependent** if there are nonzero constants $c_1$ and $c_2$ with $c_1f(t) + c_2g(t) = 0$ for all $t$. Otherwise they are called **linearly independent**.

Example

The functions $f(t) = 2\sin^2 t$ and $g(t) = 1 - \cos^2(t)$ are linearly dependent since

$$\left(1\right)(2\sin^2 t) + \left(-2\right)(1 - \cos^2(t)) = 0.$$

Example

The functions $f(t) = t$ and $g(t) = t^2$ are linearly independent since otherwise there would be nonzero constants $c_1$ and $c_2$ such that

$$c_1t + c_2t^2 = 0.$$

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for all values of \(t\). First let \(t = 1\). Then
\[
[ c_1 + c_2 = 0. \text{\nonumber}]
\]
Now let \(t = 2\). Then
\[
[ 2c_1 + 4c_2 = 0 \text{\nonumber}]
\]
This is a system of 2 equations and two unknowns. The determinant of the corresponding matrix is
\[
[4 - 2 = 2. \text{\nonumber}]
\]
Since the determinant is nonzero, the only solution is the trivial solution. That is
\[
[ c_1 = c_2 = 0 \text{\nonumber}]
\]
The two functions are linearly independent.

In the above example, we arbitrarily selected two values for \(t\). It turns out that there is a systematic way to check for linear dependence. The following theorem states this way.

Theorem

Let \(f\) and \(g\) be differentiable on \([a,b]\). If Wronskian \(W(f,g)(t_0)\) is nonzero for some \(t_0\) in \([a,b]\) then \(f\) and \(g\) are linearly independent on \([a,b]\). If \(f\) and \(g\) are linearly dependent then the Wronskian is zero for all \(t\) in \([a,b]\).

Example \(\PageIndex{3}\)

Show that the functions \(f(t) = t\) and \(g(t) = e^{2t}\) are linearly independent.

Solution

We compute the Wronskian.
\[
f'(t) = 1 g'(t) = 2e^{2t}
\]
The Wronskian is
\[
(\text{t})(2e^{2t}) - (e^{2t})(1)\nonumber
\]
Now plug in \(t=0\) to get
\[
W(f, g)(0) = -1 \text{\nonumber}
\]
which is nonzero. We can conclude that \(f\) and \(g\) are linearly independent.
Proof

If

\[ C_1 f(t) + C_2 g(t) = 0 \]

Then we can take derivatives of both sides to get

\[ C_1 f''(t) + C_2 g'(t) = 0 \]

This is a system of two equations with two unknowns. The determinant of the corresponding matrix is the Wronskian. Hence, if the Wronskian is nonzero at some \( t_0 \), only the trivial solution exists. Hence they are linearly independent.

\( \square \)

There is a fascinating relationship between second order linear differential equations and the Wronskian. This relationship is stated below.

Theorem: Abel's Theorem

Let \( y_1 \) and \( y_2 \) be solutions on the differential equation

\[ L(y) = y'' + p(t)y' + q(t)y = 0 \]

where \( p(t) \) and \( q(t) \) are continuous on \( [a,b] \). Then the Wronskian is given by

\[ W(y_1, y_2)(t) = ce^{-\int p(t) dt} \]

where \( c \) is a constant depending on only \( y_1 \) and \( y_2 \), but not on \( t \). The Wronskian is either zero for all \( t \) in \( [a,b] \) or not in \( [a,b] \).

Proof

First the Wronskian

\[ W = y_1y_2' - y_1'y_2 \]

has derivative

\[ W' = y_1y_2'' - y_1'y_2' \]

Since \( y_1 \) and \( y_2 \) are solutions to the differential equation, we have

\[ y_1'' + p(t)y_1' + q(t)y_1 = 0 \]

\[ y_2'' + p(t)y_2' + q(t)y_2 = 0 \]
Multiplying the first equation by $-y_2$ and the second by $y_1$ and adding gives

$$\frac{d}{dt}(y_1y''_2 - y''_1y_2) + p(t)(y_1y'_2 - y_1y_2) = 0.$$ 

This can be written as

$$W' + p(t)W = 0.$$ 

This is a separable differential equation with

$$\frac{dW}{W} = -p(t) dt.$$ 

Now integrate and Abel's theorem appears.

$$W(t) = ce^{\sin t}.$$

Example

Find the Wronskian (up to a constant) of the differential equations

$$y'' + \cos(t)y = 0.$$ 

Solution

We just use Abel's theorem, the integral of $\cos(t)$ is $\sin(t)$ hence the Wronskian is

$$W(t) = ce^{\sin t}.$$ 

A corollary of Abel's theorem is the following

Corollary

Let $(y_1)$ and $(y_2)$ be solutions to the differential equation

$$L(y) = y'' + p(t)y' + q(t)y = 0.$$ 

Then either $W(y_1, y_2)$ is zero for all $(t)$ or never zero.

Example

Prove that

$$y_1(t) = 1 - t \quad \text{and} \quad y_2(t) = t^3$$ 

cannot both be solutions to a differential equation.
\[ y'' + p(t)y + q(t) = 0 \]

for \( p(t) \) and \( q(t) \) continuous on \([-1, 5]\).

**Solution**

We compute the Wronskian

\[ y'_1 = -1 \quad \text{and} \quad y'_2 = 3t^2 \]

\[ W(y_1, y_2) = (1 - t)(3t^2) - (t^3)(-1) = 3t^2 - 2t^3. \]

Notice that the Wronskian is zero at \( t = 0 \) but nonzero at \( t = 1 \). By the above corollary, \( y_1 \) and \( y_2 \) cannot both be solutions.

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**Contributors**

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