3.6: Linear Independence and the Wronskian

Recall from linear algebra that two vectors \( v \) and \( w \) are called linearly dependent if there are nonzero constants \( c_1 \) and \( c_2 \) with \[ c_1v + c_2w = 0. \]

We can think of differentiable functions \( f(t) \) and \( g(t) \) as being vectors in the vector space of differentiable functions. The analogous definition is below.

Definition: Linear Dependence and Independence

Let \( f(t) \) and \( g(t) \) be differentiable functions. Then they are called \textit{linearly dependent} if there are nonzero constants \( c_1 \) and \( c_2 \) with \( c_1f(t) + c_2g(t) = 0 \) for all \( t \). Otherwise they are called \textit{linearly independent}.

Example \((\PageIndex{1})\)

The functions \( f(t) = 2\sin^2 t \) and \( g(t) = 1 - \cos^2(t) \) are linearly dependent since \[ (1)(2\sin^2 t) + (-2)(1 - \cos^2(t)) = 0. \]

Example \((\PageIndex{1})\)

The functions \( f(t) = t \) and \( g(t) = t^2 \) are linearly independent since otherwise there would be nonzero constants \( c_1 \) and \( c_2 \) such that \[ c_1t + c_2t^2 = 0 \]
for all values of \( t \). First let \( t = 1 \). Then

\[
[ c_1 + c_2 = 0 \nonumber ]
\]

Now let \( t = 2 \). Then

\[
[ 2c_1 + 4c_2 = 0 \nonumber ]
\]

This is a system of 2 equations and two unknowns. The determinant of the corresponding matrix is

\[
[ 4 - 2 = 2 \nonumber ]
\]

Since the determinant is nonzero, the only solution is the trivial solution. That is

\[
[ c_1 = c_2 = 0 \nonumber ]
\]

The two functions are linearly independent.

In the above example, we arbitrarily selected two values for \( t \). It turns out that there is a systematic way to check for linear dependence. The following theorem states this way.

**Theorem**

Let \( f \) and \( g \) be differentiable on \([a,b]\). If Wronskian \( W(f,g)(t_0) \) is nonzero for some \( t_0 \) in \([a,b]\) then \( f \) and \( g \) are linearly independent on \([a,b]\). If \( f \) and \( g \) are linearly dependent then the Wronskian is zero for all \( t \) in \([a,b]\).

**Example**

Show that the functions \( f(t) = t \) and \( g(t) = e^{2t} \) are linearly independent.

**Solution**

We compute the Wronskian.

\[
f'(t) = 1 \quad g'(t) = 2e^{2t}
\]

The Wronskian is

\[
( t)(2e^{2t}) - (e^{2t})(1) \nonumber
\]

Now plug in \( t=0 \) to get

\[
W(f, g)(0) = -1 \nonumber
\]

which is nonzero. We can conclude that \( f \) and \( g \) are *linearly independent*. 
Proof

If

\[ C_1 f(t) + C_2 g(t) = 0 \]

Then we can take derivatives of both sides to get

\[ C_1 f'(t) + C_2 g'(t) = 0 \]

This is a system of two equations with two unknowns. The determinant of the corresponding matrix is the Wronskian. Hence, if the Wronskian is nonzero at some \( t_0 \), only the trivial solution exists. Hence they are linearly independent.

\( \square \)

There is a fascinating relationship between second order linear differential equations and the Wronskian. This relationship is stated below.

Theorem: Abel's Theorem

Let \( y_1 \) and \( y_2 \) be solutions on the differential equation

\[ L(y) = y'' + p(t)y' + q(t)y = 0 \]

where \( (p) \) and \( (q) \) are continuous on \((a,b)\). Then the Wronskian is given by

\[ W(y_1, y_2)(t) = ce^{- \int p(t) dt} \]

where \( (c) \) is a constant depending on only \((y_1)\) and \((y_2)\), but not on \((t)\). The Wronskian is either zero for all \((t)\) in \((a,b)\) or not in \((a,b)\).

Proof

First the Wronskian

\[ W = y_1 y_2' - y_1' y_2 \]

has derivative

\[ W' = y_1 y_2'' - y_1'' y_2 - y_1' y_2' \]

Since \((y_1)\) and \((y_2)\) are solutions to the differential equation, we have

\[ y_1'' + p(t)y_1' + q(t)y_1 = 0 \]
\[ y_2'' + p(t)y_2' + q(t)y_2 = 0 \]
Multiplying the first equation by \(-y_2\) and the second by \(y_1\) and adding gives
\[
( y_1y''_2 - y''_1y_2) + p(t)(y_1y'_2 - y_1y_2) = 0. \nonumber\]
This can be written as
\[
W' + p(t)W = 0. \nonumber\]
This is a separable differential equation with
\[
\dfrac{dW}{W} = -p(t) dt. \nonumber\]
Now integrate and Abel's theorem appears.
\[
\square
\]
Example \(\PageIndex{4}\)
Find the Wronskian (up to a constant) of the differential equations
\[
y'' + \cos(t) y = 0. \nonumber\]
Solution
We just use Abel's theorem, the integral of \(\cos t\) is \(\sin t\) hence the Wronskian is
\[
W(t) = ce^{\sin t}. \nonumber\]
A corollary of Abel's theorem is the following
Corollary
Let \(y_1\) and \(y_2\) be solutions to the differential equation
\[
L(y) = y'' + p(t)y' + q(t)y = 0 \]
Then either \(W(y_1, y_2)\) is zero for all \(t\) or never zero.
Example \(\PageIndex{5}\)
Prove that
\[y_1(t) = 1 - t \quad \text{and } \quad y_2(t) = t^3\]
cannot both be solutions to a differential equation.
\[ y'' + p(t)y + q(t) = 0 \]

for \( p(t) \) and \( q(t) \) continuous on \([-1, 5]\).

**Solution**

We compute the Wronskian

\[ y'_1 = -1 \quad \text{and} \quad y'_2 = 3t^2 \]

\[ W(y_1, y_2) = (1 - t)(3t^2) - (t^3)(-1) = 3t^2 - 2t^3. \]

Notice that the Wronskian is zero at \( t = 0 \) but nonzero at \( t = 1 \). By the above corollary, \( y_1 \) and \( y_2 \) cannot both be solutions.

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**Contributors and Attributions**

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