Recall from linear algebra that two vectors \( \langle v \rangle \) and \( \langle w \rangle \) are called linearly dependent if there are nonzero constants \( c_1 \) and \( c_2 \) with
\[
 c_1v + c_2w = 0.
\]

We can think of differentiable functions \( \langle f(t) \rangle \) and \( \langle g(t) \rangle \) as being vectors in the vector space of differentiable functions. The analogous definition is below.

**Definition: Linear Dependence and Independence**

Let \( f(t) \) and \( g(t) \) be differentiable functions. Then they are called *linearly dependent* if there are nonzero constants \( c_1 \) and \( c_2 \) with
\[
 c_1f(t) + c_2g(t) = 0
\]
for all \( t \). Otherwise they are called *linearly independent*.

**Example**

The functions \( f(t) = 2\sin^2 t \) and \( g(t) = 1 - \cos^2(t) \) are linearly dependent since
\[
 (1)(2\sin^2 t) + (-2)(1 - \cos^2(t)) = 0.
\]

**Example**

The functions \( f(t) = t \) and \( g(t) = t^2 \) are linearly independent since otherwise there would be nonzero constants \( c_1 \) and \( c_2 \) such that
\[
 c_1t + c_2t^2 = 0
\]
for all values of \( t \). First let \( t = 1 \). Then
\[
\begin{align*}
\text{c}_1 + \text{c}_2 &= 0. \\
\text{c}_1 + 2\text{c}_2 &= 0
\end{align*}
\]
Now let \( t = 2 \). Then
\[
\begin{align*}
2\text{c}_1 + 4\text{c}_2 &= 0
\end{align*}
\]
This is a system of 2 equations and two unknowns. The determinant of the corresponding matrix is
\[
\begin{align*}
4 - 2 &= 2
\end{align*}
\]
Since the determinant is nonzero, the only solution is the trivial solution. That is
\[
\begin{align*}
\text{c}_1 = \text{c}_2 = 0
\end{align*}
\]
The two functions are linearly independent.

In the above example, we arbitrarily selected two values for \( t \). It turns out that there is a systematic way to check for linear dependence. The following theorem states this way.

**Theorem**

Let \( f \) and \( g \) be differentiable on \( [a,b] \). If Wronskian \( W(f,g)(t_0) \) is nonzero for some \( t_0 \) in \( [a,b] \) then \( f \) and \( g \) are linearly independent on \( [a,b] \). If \( f \) and \( g \) are linearly dependent then the Wronskian is zero for all \( t \) in \( [a,b] \).

**Example \( \PageIndex{3} \)**

Show that the functions \( f(t) = t \) and \( g(t) = e^{2t} \) are linearly independent.

**Solution**

We compute the Wronskian.
\[
\begin{align*}
\text{f}'(t) &= 1 \\
\text{g}'(t) &= 2e^{2t}
\end{align*}
\]
The Wronskian is
\[
\begin{align*}
\text{W}(f, g)(t) &= (t)(2e^{2t}) - (e^{2t})(1)
\end{align*}
\]
Now plug in \( t=0 \) to get
\[
\text{W}(f, g)(0) = -1
\]
which is nonzero. We can conclude that \( f \) and \( g \) are linearly independent.
Proof

If

\[
C_1 f(t) + C_2 g(t) = 0
\]

Then we can take derivatives of both sides to get

\[
C_1 f'(t) + C_2 g'(t) = 0
\]

This is a system of two equations with two unknowns. The determinant of the corresponding matrix is the Wronskian. Hence, if the Wronskian is nonzero at some \( t_0 \), only the trivial solution exists. Hence they are linearly independent.

\( \square \)

There is a fascinating relationship between second order linear differential equations and the Wronskian. This relationship is stated below.

Theorem: Abel's Theorem

Let \( y_1 \) and \( y_2 \) be solutions on the differential equation

\[
L(y) = y'' + p(t)y' + q(t)y = 0
\]

where \( p \) and \( q \) are continuous on \([a,b]\). Then the Wronskian is given by

\[
W(y_1, y_2)(t) = ce^{-\int p(t) \, dt}
\]

where \( c \) is a constant depending on only \( y_1 \) and \( y_2 \), but not on \( t \). The Wronskian is either zero for all \( t \) in \([a,b]\) or not in \([a,b]\).

Proof

First the Wronskian

\[
W = y_1 y_2' - y_1'y_2
\]

has derivative

\[
W' = y_1 y_2'' + y_1'y_2'' - y_1 y_2'' = y_1 y_2'' + y_1 y_2'' - y_1 y_2'' - y_1 y_2'' = y_1 y_2'' - y_1 y_2''.
\]

Since \( y_1 \) and \( y_2 \) are solutions to the differential equation, we have

\[
y_1'' + p(t)y_1' + q(t)y_1 = 0
\]

\[
y_2'' + p(t)y_2' + q(t)y_2 = 0
\]
Multiplying the first equation by \(-y_2\) and the second by \(y_1\) and adding gives

\[
(y_1y''_2 - y''_1y_2) + p(t)(y_1y'_2 - y_1y_2) = 0. \nonumber
\]

This can be written as

\[
W' + p(t)W = 0. \nonumber
\]

This is a separable differential equation with

\[
\frac{dW}{W} = -p(t) \, dt. \nonumber
\]

Now integrate and Abel's theorem appears.

\(\square\)

Example \(\PageIndex{4}\)

Find the Wronskian (up to a constant) of the differential equations

\[
y'' + \cos(t) \, y = 0. \nonumber
\]

Solution

We just use Abel's theorem, the integral of \(\cos t\) is \(\sin t\) hence the Wronskian is

\[
W(t) = ce^{\sin t}. \nonumber
\]

A corollary of Abel's theorem is the following

Corollary

Let \(\ y_1\) and \(\ y_2\) be solutions to the differential equation

\[
L(y) = y'' + p(t)y' + q(t)y = 0 \]

Then either \(W(y_1, y_2)\) is zero for all \(t\) or never zero.

Example \(\PageIndex{5}\)

Prove that

\[
y_1(t) = 1 - t \quad \text{and} \quad y_2(t) = t^3
\]

cannot both be solutions to a differential equation.
\[ y'' + p(t)y + q(t) = 0 \]

for \( p(t) \) and \( q(t) \) continuous on \( \left[-1, 5\right] \).

**Solution**

We compute the Wronskian

\[
y'_1 = -1 \quad \text{and} \quad y'_2 = 3t^2
\]

\[
W(y_1, y_2) = (1 - t)(3t^2) - (t^3)(-1) = 3t^2 - 2t^3.
\]

Notice that the Wronskian is zero at \( t = 0 \) but nonzero at \( t = 1 \). By the above corollary, \( y_1 \) and \( y_2 \) cannot both be solutions.

**Contributors and Attributions**

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