3.6: Linear Independence and the Wronskian

Recall from linear algebra that two vectors \((v)\) and \((w)\) are called linearly dependent if there are nonzero constants \(c_1\) and \(c_2\) with

\[
\[ c_1v + c_2w = 0. \]
\]

We can think of differentiable functions \((f(t))\) and \((g(t))\) as being vectors in the vector space of differentiable functions. The analogous definition is below.

Definition: Linear Dependence and Independence

Let \((f(t))\) and \((g(t))\) be differentiable functions. Then they are called **linearly dependent** if there are nonzero constants \(c_1\) and \(c_2\) with \(c_1f(t) + c_2g(t) = 0\) for all \(t\). Otherwise they are called **linearly independent**.

Example \(\PageIndex{1}\)

The functions \((f(t) = 2\sin^2 t)\) and \((g(t) = 1 - \cos^2(t))\) are linearly dependent since

\[
\[(1)(2\sin^2 t) + (-2)(1 - \cos^2(t)) = 0. \nonumber\]
\]

Example \(\PageIndex{1}\)

The functions \((f(t) = t)\) and \((g(t) = t^2)\) are linearly independent since otherwise there would be nonzero constants \(c_1\) and \(c_2\) such that

\[
\[ c_1t + c_2t^2 = 0 \nonumber\]
\]
for all values of \( t \). First let \( t = 1 \). Then

\[
\begin{align*}
&c_1 + c_2 = 0. \\
\end{align*}
\]

Now let \( t = 2 \). Then

\[
\begin{align*}
&2c_1 + 4c_2 = 0
\end{align*}
\]

This is a system of 2 equations and two unknowns. The determinant of the corresponding matrix is

\[
[4 - 2 = 2]
\]

Since the determinant is nonzero, the only solution is the trivial solution. That is

\[
\begin{align*}
&c_1 = c_2 = 0
\end{align*}
\]

The two functions are linearly independent.

In the above example, we arbitrarily selected two values for \( t \). It turns out that there is a systematic way to check for linear dependence. The following theorem states this way.

**Theorem**

Let \( f \) and \( g \) be differentiable on \( [a,b] \). If Wronskian \( (W(f,g)(t_0)) \) is nonzero for some \( t_0 \) in \( [a,b] \) then \( f \) and \( g \) are linearly independent on \( [a,b] \). If \( f \) and \( g \) are linearly dependent then the Wronskian is zero for all \( t \) in \( [a,b] \).

**Example \( \PageIndex{3} \)**

Show that the functions \( f(t) = t \) and \( g(t) = e^{2t} \) are linearly independent.

**Solution**

We compute the Wronskian.

\[
\begin{align*}
&f'(t) = 1 \quad g'(t) = 2e^{2t}
\end{align*}
\]

The Wronskian is

\[
\begin{align*}
&\begin{vmatrix}
 t & 2e^{2t} \\
 e^{2t} & \end{vmatrix}
\end{align*}
\]

Now plug in \( t=0 \) to get

\[
\begin{align*}
&W(f, g)(0) = -1
\end{align*}
\]

which is nonzero. We can conclude that \( f \) and \( g \) are linearly independent.
Proof

If

\[ C_1 f(t) + C_2 g(t) = 0 \]

Then we can take derivatives of both sides to get

\[ C_1 f''(t) + C_2 g'(t) = 0 \]

This is a system of two equations with two unknowns. The determinant of the corresponding matrix is the Wronskian. Hence, if the Wronskian is nonzero at some \( t_0 \), only the trivial solution exists. Hence they are linearly independent.

\( \square \)

There is a fascinating relationship between second order linear differential equations and the Wronskian. This relationship is stated below.

Theorem: Abel's Theorem

Let \( y_1 \) and \( y_2 \) be solutions on the differential equation

\[ L(y) = y'' + p(t)y' + q(t)y = 0 \]

where \( p(t) \) and \( q(t) \) are continuous on \( [a,b] \). Then the Wronskian is given by

\[ W(y_1, y_2)(t) = ce^{-\int p(t)\, dt} \]

where \( c \) is a constant depending on only \( y_1 \) and \( y_2 \), but not on \( t \). The Wronskian is either zero for all \( t \) in \( [a,b] \) or not in \( [a,b] \).

Proof

First the Wronskian

\[ W = y'_1 y''_2 - y'_2 y''_1 \]

has derivative

\[ W' = y'_1 y''_2 - y'_2 y''_1 \]

Since \( y_1 \) and \( y_2 \) are solutions to the differential equation, we have

\[ y''_1 + p(t)y'_1 + q(t)y_1 = 0 \]
\[ y''_2 + p(t)y'_2 + q(t)y_2 = 0 \]
Multiplying the first equation by \(-y_2\) and the second by \(y_1\) and adding gives

\[
(y_1 y''_2 - y''_1 y_2) + p(t)(y_1 y'_2 - y_1 y_2) = 0. \nonumber
\]

This can be written as

\[
W' + p(t)W = 0. \nonumber
\]

This is a separable differential equation with

\[
\dfrac{dW}{W} = -p(t) \, dt. \nonumber
\]

Now integrate and Abel's theorem appears.

Example \(\PageIndex{4}\)

Find the Wronskian (up to a constant) of the differential equations

\[
y'' + \cos(t) \, y = 0. \nonumber
\]

Solution

We just use Abel's theorem, the integral of \(\cos t\) is \(\sin t\) hence the Wronskian is

\[
W(t) = ce^{\sin t}. \nonumber
\]

A corollary of Abel's theorem is the following

Corollary

Let \(y_1\) and \(y_2\) be solutions to the differential equation

\[
L(y) = y'' + p(t)y' + q(t)y = 0 \]

Then either \(W(y_1, y_2)\) is zero for all \(t\) or never zero.

Example \(\PageIndex{5}\)

Prove that

\[
y_1(t) = 1 - t \quad \text{and} \quad y_2(t) = t^3
\]

cannot both be solutions to a differential equation
\[ y'' + p(t)y + q(t) = 0 \] for \( p(t) \) and \( q(t) \) continuous on \( \left[-1, 5\right] \).

**Solution**

We compute the Wronskian

\[ y'_1 = -1 \quad \text{and} \quad y'_2 = 3t^2 \]

\[ W(y_1, y_2) = (1 - t)(3t^2) - (t^3)(-1) = 3t^2 - 2t^3. \]

Notice that the Wronskian is zero at \( t = 0 \) but nonzero at \( t = 1 \). By the above corollary, \( y_1 \) and \( y_2 \) cannot both be solutions.

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