8.2: Infinite Series

Given the sequence \( \{a_n\} = \{1/2^n\} = 1/2, 1/4, 1/8, \ldots \), consider the following sums:

\[
\begin{array}{cccccc}
a_1 &=& 1/2 &=& 1/2 \\
a_1+a_2 &=& 1/2+1/4 &=& 3/4 \\
a_1+a_2+a_3 &=& 1/2+1/4+1/8 &=& 7/8 \\
a_1+a_2+a_3+a_4 &=& 1/2+1/4+1/8+1/16 &=& 15/16 \\
\end{array}
\]

In general, we can show that

\[
a_1+a_2+a_3+\cdots+a_n = \frac{2^n-1}{2^n} = 1-\frac{1}{2^n}.
\]

Let \( S_n \) be the sum of the first \( n \) terms of the sequence \( \{1/2^n\} \). From the above, we see that \( S_1 = 1/2 \), \( S_2 = 3/4 \), etc. Our formula at the end shows that \( S_n = 1-1/2^n \).

Now consider the following limit:

\[
\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1-\frac{1}{2^n}\right) = 1.
\]

This limit can be interpreted as saying something amazing: the sum of all the terms of the sequence \( \{1/2^n\} \) is 1. This example illustrates some interesting concepts that we explore in this section. We begin this exploration with some definitions.

Definition 31: Infinite Series, \( n^{\text{th}} \) Partial Sums, Convergence, Divergence
Let \(\{a_n\}\) be a sequence.

1. The sum \(\sum_{n=1}^{\infty} a_n\) is an **infinite series** (or, simply **series**).
2. Let \(S_n = \sum_{i=1}^{n} a_i\); the sequence \(\{S_n\}\) is the sequence of \((n^{th})\) **partial sums** of \(\{a_n\}\).
3. If the sequence \(\{S_n\}\) converges to \(L\), we say the series \(\sum_{n=1}^{\infty} a_n\) **converges** to \(L\), and we write \(\sum_{n=1}^{\infty} a_n = L\).
4. If the sequence \(\{S_n\}\) diverges, the series \(\sum_{n=1}^{\infty} a_n\) **diverges**.

Using our new terminology, we can state that the series \(\sum_{n=1}^{\infty} \frac{1}{2^n}\) converges, and \(\sum_{n=1}^{\infty} \frac{1}{2^n} = 1\).

We will explore a variety of series in this section. We start with two series that diverge, showing how we might discern divergence.

**Example \(\PageIndex{1}\): Showing series diverge**

1. Let \(\{a_n\} = \{n^2\}\). Show \(\sum_{n=1}^{\infty} a_n\) diverges.
2. Let \(\{b_n\} = \{(-1)^{n+1}\}\). Show \(\sum_{n=1}^{\infty} b_n\) diverges.

**Solution**

1. Consider \(S_n\), the \((n^{th})\) partial sum. \[
S_n = a_1 + a_2 + a_3 + \cdots + a_n
\]
By Theorem 37, this is \[
\frac{n(n+1)(2n+1)}{6}.
\] Since \(\lim_{n\to\infty}S_n = \infty\), we conclude that the series \(\sum_{n=1}^{\infty} n^2\) diverges. It is instructive to write \(\sum_{n=1}^{\infty} n^2 = \infty\) for this tells us how the series diverges: it grows without bound.

A scatter plot of the sequences \(\{a_n\}\) and \(\{S_n\}\) is given in Figure 8.7(a). The terms of \(\{a_n\}\) are growing, so the terms of the partial sums \(\{S_n\}\) are growing even faster, illustrating that the series diverges.

2. The sequence \(\{b_n\}\) starts with 1, \((-1)\), 1, \((-1)\), \((\ldots)\). Consider some of the partial sums \(\{S_n\}\) of \(\{b_n\}\): \[
S_1 = 1\]
\(S_2 = 0\)
\(S_3 = 1\)
\(S_4 = 0\)
This pattern repeats; we find that \(S_n = \left\{\begin{array}{ll} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{array}\right\}\) for this tells us how the series diverges: it grows without bound.

As \(\{S_n\}\) oscillates, repeating 1, 0, 1, 0, \((\ldots)\), we conclude that \(\lim_{n\to\infty}S_n\) does not exist, hence \(\sum_{n=1}^{\infty} (-1)^{n+1}\) diverges.

A scatter plot of the sequence \(\{b_n\}\) and the partial sums \(\{S_n\}\) is given in Figure 8.7(b). When \(n\) is odd, \(b_n = S_n\) so the marks for \(b_n\) are drawn oversize to show they coincide.
While it is important to recognize when a series diverges, we are generally more interested in the series that converge. In this section we will demonstrate a few general techniques for determining convergence; later sections will delve deeper into this topic.

**Geometric Series**

One important type of series is a *geometric series*.

Definition 32: geometric series
A geometric series is a series of the form
\[
\sum_{n=0}^\infty r^n = 1+r+r^2+r^3+\cdots+r^n+\cdots
\]
Note that the index starts at \(n=0\), not \(n=1\).

We started this section with a geometric series, although we dropped the first term of \(1\). One reason geometric series are important is that they have nice convergence properties.

**Theorem 60: Convergence of geometric series**

Consider the geometric series \( \sum_{n=0}^\infty r^n \).

1. The \(n\text{th}\) partial sum is: \( S_n = \frac{1-r^{n+1}}{1-r} \).

2. The series converges if, and only if, \(|r| < 1\). When \(|r|<1\),
\[
\sum_{n=0}^\infty r^n = \frac{1}{1-r}.
\]

According to Theorem 60, the series
\[
\sum_{n=0}^\infty \frac{1}{2^n} = \sum_{n=0}^\infty \left(\frac{1}{2}\right)^n = 1+\frac{1}{2}+\frac{1}{4}+\cdots
\]
converges as \(r=1/2\), and \(\sum_{n=0}^\infty \frac{1}{2^n} = \frac{1}{1-1/2} = 2\). This concurs with our introductory example; while there we got a sum of 1, we skipped the first term of 1.

**Example \(\PageIndex{2}\): Exploring geometric series**

Check the convergence of the following series. If the series converges, find its sum.

\(1. \sum_{n=2}^\infty \left(\frac{3}{4}\right)^n \quad 2. \sum_{n=0}^\infty \left(\frac{-1}{2}\right)^n \quad 3. \sum_{n=0}^\infty 3^n\)

**Solution**
1. Since \( r = 3/4 < 1 \), this series converges. By Theorem 60, we have that \[ \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{1-\frac{3}{4}} = 4. \] However, note the subscript of the summation in the given series: we are to start with \( n=2 \). Therefore we subtract off the first two terms, giving: \[ \sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n = 4 - 1 - \frac{3}{4} = \frac{9}{4}. \] This is illustrated in Figure 8.8.

2. Since \( |r| = 1/2 < 1 \), this series converges, and by Theorem 60, \[ \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}. \] The partial sums of this series are plotted in Figure 8.9(a). Note how the partial sums are not purely increasing as some of the terms of the sequence \( \{-(-1/2)^n\} \) are negative.

3. Since \( r > 1 \), the series diverges. (This makes "common sense"; we expect the sum \( 1+3+9+27 + 81+243+\cdots \) to diverge.) This is illustrated in Figure 8.9(b).
Another important type of series is the *p-series*.

**Definition 33:** \((p)\)-Series, General \((P)\)-Series

1. A \((p)\)--series is a series of the form \[\sum_{n=1}^{\infty} \frac{1}{n^p}, \text{ where } p > 0.\]
2. A general \((p)\)--series is a series of the form \[\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}, \text{ where } p > 0 \text{ and } a, b \text{ are real numbers.}\]

Like geometric series, one of the nice things about \(p\)--series is that they have easy to determine convergence properties.

**Theorem 61:** convergence of general \((P)\)--Series
A general \((p)\)--series \(\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}\) will converge if, and only if, \((p>1)\).

**Note:** Theorem 61 assumes that \((an+b\neq 0)\) for all \((n)\). If \((an+b=0)\) for some \((n)\), then of course the series does not converge regardless of \((p)\) as not all of the terms of the sequence are defined.

Example \(\PageIndex{3}\): Determining convergence of series

Determine the convergence of the following series.

1. \[\sum_{n=1}^{\infty} \frac{1}{n}\]
2. \[\sum_{n=1}^{\infty} \frac{1}{n^2}\]
3. \[\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\]
4. \[\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\]
5. \[\sum_{n=11}^{\infty} \frac{1}{(\frac{1}{2}n-5)^3}\]
6. \[\sum_{n=1}^{\infty} \frac{1}{2^n}\]

**Solution**

1. This is a \((p)\)--series with \((p=1)\). By Theorem 61, this series diverges. This series is a famous series, called the Harmonic Series, so named because of its relationship to harmonics in the study of music and sound.
2. This is a \((p)\)--series with \((p=2)\). By Theorem 61, it converges. Note that the theorem does not give a formula by which we can determine what the series converges to; we just know it converges. A famous, unexpected result is that this series converges to \(\frac{\pi^2}{6}\).
3. This is a \((p)\)--series with \((p=1/2)\); the theorem states that it diverges.
4. This is not a \((p)\)--series; the definition does not allow for alternating signs. Therefore we cannot apply Theorem 61. (Another famous result states that this series, the Alternating Harmonic Series, converges to \(\ln 2\).)
5. This is a general \((p)\)--series with \((p=3)\), therefore it converges.
6. This is not a \((p)\)--series, but a geometric series with \((r=1/2)\). It converges.

Later sections will provide tests by which we can determine whether or not a given series converges. This, in general, is much easier than determining what a given series converges to. There are many cases, though, where the sum can be determined.

Example \(\PageIndex{4}\): Telescoping series

Evaluate the sum \(\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)\).

**Solution**

It will help to write down some of the first few partial sums of this series.

\[
\begin{align*}
S_1 &= \frac{1}{1} - \frac{1}{2} & &= 1 - \frac{1}{2} \\
S_2 &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) & &= 1 - \frac{1}{3} \\
S_3 &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) & &= 1 - \frac{1}{4} \\
\end{align*}
\]
Note how most of the terms in each partial sum are canceled out! In general, we see that \( S_n = 1 - \frac{1}{n+1} \). The sequence \( \{ S_n \} \) converges, as \( \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left( 1 - \frac{1}{n+1} \right) = 1 \), and so we conclude that \( \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = 1 \). Partial sums of the series are plotted in Figure 8.10.

The series in Example 8.2.4 is an example of a **telescoping series**. Informally, a telescoping series is one in which the partial sums reduce to just a finite number of terms. The partial sum \( S_n \) did not contain \( n \) terms, but rather just two: 1 and \( 1/(n+1) \).

When possible, seek a way to write an explicit formula for the \( n^{\text{th}} \) partial sum \( S_n \). This makes evaluating the limit \( \lim_{n \to \infty} S_n \) much more approachable. We do so in the next example.

**Note on notation:** Most of the series we encounter will start with \( n=1 \). For ease of notation, we will often write \( \sum_{n=1}^{\infty} a_n \) instead of writing \( \sum_{n=1}^{\infty} a_n \).

**Example:** Evaluating series

Evaluate each of the following infinite series.

1. \( \sum_{n=1}^{\infty} \frac{2}{n^2+2n} \)
2. \( \sum_{n=1}^{\infty} \ln \left( \frac{n+1}{n} \right) \)

**Solution**
1. We can decompose the fraction \(2/(n^2+2n)\) as \(\frac{2}{n^2+2n} = \frac{1}{n} - \frac{1}{n+2}\). (See Section 6.5, Partial Fraction Decomposition, to recall how this is done, if necessary.)

Expressing the terms of \(\{S_n\}\) is now more instructive:

\[
\begin{align*}
S_1 &= 1 - \frac{1}{3} &&= 1 - \frac{1}{3} \\
S_2 &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) &&= 1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} \\
S_3 &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) &&= 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} \\
S_4 &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) &&= 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6} \\
S_5 &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) &&= 1 + \frac{1}{2} - \frac{1}{6} - \frac{1}{7} \\
\end{align*}
\]

We again have a telescoping series. In each partial sum, most of the terms cancel and we obtain the formula \(S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\). Taking limits allows us to determine the convergence of the series: \(\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{2}\). Therefore \(\sum_{n=1}^{\infty} \frac{1}{n^2+2n} = \frac{3}{2}\).

This is illustrated in Figure 8.11(a).

2. We begin by writing the first few partial sums of the series:

\[
\begin{align*}
S_1 &= \ln(2) \\
S_2 &= \ln(2) + \ln\left(\frac{3}{2}\right) \\
S_3 &= \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) \\
S_4 &= \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right)
\end{align*}
\]

At first, this does not seem helpful, but recall the logarithmic identity: \(\ln x + \ln y = \ln(xy)\). Applying this to \(S_4\) gives: \[S_4 = \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right) = \ln\left(\frac{2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}}{1}\right) = \ln(5).\]

We can conclude that \(\{S_n\} \to \ln(n+1)\). This sequence does not converge, as \(\lim_{n \to \infty} \ln(n+1) = \infty\). Therefore \(\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \infty\); the series diverges. Note in Figure 8.11(b) how the sequence of partial sums grows slowly; after 100 terms, it is not yet over 5. Graphically we may be fooled into thinking the series converges, but our analysis above shows that it does not.
Figure 8.11: Scatter plots relating to the series in Example 8.2.5

We are learning about a new mathematical object, the series. As done before, we apply "old" mathematics to this new topic.

THEOREM 62 PROPERTIES OF INFINITE SERIES

Let \( \sum_{n=1}^{\infty} a_n = L \), \( \sum_{n=1}^{\infty} b_n = K \), and let \( c \) be a constant.

1. Constant Multiple Rule: \( \sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n = c \cdot L \).

2. Sum/Difference Rule: \( \sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = L \pm K \).

Before using this theorem, we provide a few "famous" series.
KEY IDEA 31 IMPORTANT SERIES

1. \( \sum_{n=0}^{\infty} \frac{1}{n!} = e \). (Note that the index starts with \( n=0 \).)
2. \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \).
3. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \).
4. \( \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} \).
5. \( \sum_{n=1}^{\infty} \frac{1}{n} \) \quad (This is called the Harmonic Series.)
6. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2 \). (This is called the Alternating Harmonic Series.)

Example \( \PageIndex{6} \): Evaluating series

Evaluate the given series.

\[ \begin{align*}
1. \quad & \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^2-n)}{n^3} \\
&= \sum_{n=1}^{\infty} \left( \frac{(-1)^{n+1}n^2}{n^3} - \frac{(-1)^{n+1}n}{n^3} \right) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \\
&= \ln(2) - \frac{\pi^2}{12} \approx -0.1293.
\end{align*} \]
This is illustrated in Figure 8.12(a).

2. This looks very similar to the series that involves \( e \) in Key Idea 31. Note, however, that the series given in this example starts with \( (n=1) \) and not \( (n=0) \). The first term of the series in the Key Idea is \( (1/0! = 1) \), so we will subtract this from our result below:

\[ \begin{align*}
&= \sum_{n=1}^{\infty} \frac{1000}{n!} = 1000 \cdot \left( \sum_{n=1}^{\infty} \frac{1}{n!} \right) = 1000 \cdot (e-1) \approx 1718.28.
\end{align*} \]
This is illustrated in Figure 8.12(b). The graph shows how this particular series converges very rapidly.

3. The denominators in each term are perfect squares; we are adding \( \sum_{n=4}^{\infty} \frac{1}{n^2} \) (note we start with \( n=4 \)), not \( n=1 \). This series will converge. Using the formula from Key Idea 31, we have the following:

\[ \begin{align*}
&= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{3} \frac{1}{n^2} \approx 0.2838.
\end{align*} \]
It may take a while before one is comfortable with this statement, whose truth lies at the heart of the study of infinite series: *it is possible that the sum of an infinite list of nonzero numbers is finite.* We have seen this repeatedly in this section, yet it still may "take some getting used to."

As one contemplates the behavior of series, a few facts become clear.

1. In order to add an infinite list of nonzero numbers and get a finite result, "most" of those numbers must be "very near" 0.
2. If a series diverges, it means that the sum of an infinite list of numbers is not finite (it may approach \(\pm \infty\) or it may oscillate), and:
1. The series will still diverge if the first term is removed.
2. The series will still diverge if the first 10 terms are removed.
3. The series will still diverge if the first \(1,000,000\) terms are removed.
4. The series will still diverge if any finite number of terms from anywhere in the series are removed.

These concepts are very important and lie at the heart of the next two theorems.

**Theorem 63** \(n^{th}\)-Term Test for Convergence/Divergence

Consider the series \(\sum_{n=1}^{\infty} a_n\).

1. If \(\sum_{n=1}^{\infty} a_n\) converges, then \(\lim_{n \to \infty} a_n = 0\).
2. If \(\lim_{n \to \infty} a_n \neq 0\), then \(\sum_{n=1}^{\infty} a_n\) diverges.

Note that the two statements in Theorem 63 are really the same. In order to converge, the limit of the terms of the sequence must approach 0; if they do not, the series will not converge.

Looking back, we can apply this theorem to the series in Example 8.2.1. In that example, the \(n^{th}\) terms of both sequences do not converge to 0, therefore we can quickly conclude that each series diverges.

**Important!** This theorem does not state that if \(\lim_{n \to \infty} a_n = 0\) then \(\sum_{n=1}^{\infty} a_n\) converges. The standard example of this is the Harmonic Series, as given in Key Idea 31. The Harmonic Sequence, \(\{1/n\}\), converges to 0; the Harmonic Series, \(\sum_{n=1}^{\infty} 1/n\), diverges.

**Theorem 64** Infinite Nature of Series

The convergence or divergence remains unchanged by the addition or subtraction of any finite number of terms. That is:

1. A divergent series will remain divergent with the addition or subtraction of any finite number of terms.
2. A convergent series will remain convergent with the addition or subtraction of any finite number of terms. (Of course, the sum will likely change.)

Consider once more the Harmonic Series \(\sum_{n=1}^{\infty} \frac{1}{n}\) which diverges; that is, the sequence of partial sums \(\{S_n\}\) grows (very, very slowly) without bound. One might think that by removing the "large" terms of the sequence that perhaps the series will converge. This is simply not the case. For instance, the sum of the first 10 million terms of the Harmonic Series is about 16.7. Removing the first 10 million terms from the Harmonic Series changes the \(\sum_{n=1}^{\infty} \frac{1}{n}\) partial sums, effectively subtracting 16.7 from the sum. However, a sequence that is growing without bound will still grow without bound when 16.7 is subtracted from it.

The equations below illustrate this. The first line shows the infinite sum of the Harmonic Series split into the sum of the first 10 million terms plus the sum of "everything else." The next equation shows us subtracting these first 10 million terms from both sides. The final equation employs a bit of "pseudo--math": subtracting 16.7 from "infinity" still leaves one with "infinity."

\[
\begin{align*}
\sum_{n=1}^{\infty} \frac{1}{n} &= \sum_{n=1}^{10,000,000} \frac{1}{n} + \sum_{n=10,000,001}^{\infty} \frac{1}{n} \\
\sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{10,000,000} \frac{1}{n} &= \sum_{n=10,000,001}^{\infty} \frac{1}{n} \\
\sum_{n=1}^{\infty} \frac{1}{n} &= \sum_{n=10,000,000}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{1}{n} - 16.7
\end{align*}
\]
This section introduced us to series and defined a few special types of series whose convergence properties are well known: we know when a \( (p) \)-series or a geometric series converges or diverges. Most series that we encounter are not one of these types, but we are still interested in knowing whether or not they converge. The next three sections introduce tests that help us determine whether or not a given series converges.

**Contributors and Attributions**

- Gregory Hartman (Virginia Military Institute). Contributions were made by Troy Siemers and Dimplekumar Chalishajar of VMI and Brian Heinold of Mount Saint Mary's University. This content is copyrighted by a Creative Commons Attribution - Noncommercial (BY-NC) License. [http://www.apexcalculus.com/](http://www.apexcalculus.com/)