9.3: Calculus and Parametric Equations

The previous section defined curves based on parametric equations. In this section we'll employ the techniques of calculus to study these curves. We are still interested in lines tangent to points on a curve. They describe how the \(y\)-values are changing with respect to the \(x\)-values, they are useful in making approximations, and they indicate instantaneous direction of travel.

The slope of the tangent line is still \(\frac{dy}{dx}\), and the Chain Rule allows us to calculate this in the context of parametric equations. If \(x=f(t)\) and \(y=g(t)\), the Chain Rule states that

\[
\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.\]

Solving for \(\frac{dy}{dx}\), we get

\[
\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{dx/dt} = \frac{g'(t)}{f'(t)},
\]

provided that \(f'(t) \neq 0\). This is important so we label it a Key Idea.

key idea 37 Finding \(\frac{dy}{dx}\) with Parametric Equations.

Let \(x=f(t)\) and \(y=g(t)\), where \(f'\) and \(g'\) are differentiable on some open interval \(I\) and \(f'(t) \neq 0\) on \(I\). Then

\[
\frac{dy}{dx} = \frac{g'(t)}{f'(t)}.
\]

We use this to define the tangent line.
Definition 47 Tangent and Normal Lines

Let a curve \( \gamma(C) \) be parametrized by \((x=f(t)), (y=g(t))\), where \((f(t))\) and \((g(t))\) are differentiable functions on some interval \((I)\) containing \((t=t_0)\). The tangent line to \((C)\) at \((t=t_0)\) is the line through \((\big(f(t_0),g(t_0)\big))\) with slope \((m=g'(t_0)/f'(t_0))\), provided \((f'(t_0)\neq 0)\).

The normal line to \((C)\) at \((t=t_0)\) is the line through \((\big(f(t_0),g(t_0)\big))\) with slope \((m=-f'(t_0)/g'(t_0))\), provided \((g'(t_0)\neq 0)\).

The definition leaves two special cases to consider. When the tangent line is horizontal, the normal line is undefined by the above definition as \((g'(t_0)=0)\). Likewise, when the normal line is horizontal, the tangent line is undefined. It seems reasonable that these lines be defined (one can draw a line tangent to the "right side" of a circle, for instance), so we add the following to the above definition.

1. If the tangent line at \((t=t_0)\) has a slope of 0, the normal line to \((C)\) at \((t=t_0)\) is the line \((x=f(t_0))\).
2. If the normal line at \((t=t_0)\) has a slope of 0, the tangent line to \((C)\) at \((t=t_0)\) is the line \((x=f(t_0))\).

Example \(\PageIndex{1}\): Tangent and Normal Lines to Curves

Let \((x=5t^2-6t+4)\) and \((y=t^2+6t-1)\), and let \((C)\) be the curve defined by these equations.

1. Find the equations of the tangent and normal lines to \((C)\) at \((t=3)\).
2. Find where \((C)\) has vertical and horizontal tangent lines.

Solution

1. We start by computing \((f'(t)=10t-6)\) and \((g'(t)=2t+6)\). Thus \((\frac{dy}{dx}=\frac{2t+6}{10t-6})\).

Make note of something that might seem unusual: \((\frac{dy}{dx})\) is a function of \((t)\), not \((x)\). Just as points on the curve are found in terms of \((t)\), so are the slopes of the tangent lines.

The point on \((C)\) at \((t=3)\) is \((((31,26))\)). The slope of the tangent line is \((m=1/2)\) and the slope of the normal line is \((m=-2)\). Thus,

- the equation of the tangent line is \((y=\frac{1}{2}(x-31)+26)\), and
- the equation of the normal line is \((y=-2(x-31)+26)\).

This is illustrated in Figure 9.29.

2. To find where \((C)\) has a horizontal tangent line, we set \((\frac{dy}{dx}=0)\) and solve for \((t)\). In this case, this amounts to setting \((g'(t)=0)\) and solving for \((t)\) (and making sure that \((f'(t)\neq 0))\).

\((g'(t)=0) \quad \Rightarrow \quad 2t+6=0 \quad \Rightarrow \quad t=-3.\)

The point on \((C)\) corresponding to \((t=-3)\) is \((((67,-10))\)); the tangent line at that point is horizontal (hence with equation \((y=-10))\).

To find where \((C)\) has a vertical tangent line, we find where it has a horizontal normal line, and set \((-\frac{1}{f'(t)})(g'(t)=0)\). This amounts to setting \((f'(t)=0)\) and solving for \((t)\) (and making sure that \((g'(t)\neq 0)))\).

\((f'(t)=0) \quad \Rightarrow \quad 10t-6=0 \quad \Rightarrow \quad t=0.6.\)

The point on \((C)\) corresponding to \((t=0.6)\) is \(((2.2,2.96))\). The tangent line at that point is \((x=2.2))\).
The points where the tangent lines are vertical and horizontal are indicated on the graph in Figure 9.29.

![Figure 9.29: Graphing tangent and normal lines in Example 9.3.1](image)

**Example 9.3.1:** Tangent and Normal Lines to a Circle

1. Find where the unit circle, defined by \(x = \cos t\) and \(y = \sin t\) on \([0, 2\pi]\), has vertical and horizontal tangent lines.
2. Find the equation of the normal line at \(t = t_0\).

**Solution**

1. We compute the derivative following Key Idea 37:
   \[
   \frac{dy}{dx} = \frac{g'(t)}{f'(t)} = -\frac{\cos t}{\sin t}.
   \]
   The derivative is \(0\) when \(\cos t = 0\); that is, when \(t = \pi/2, 3\pi/2\). These are the points \((0, 1)\) and \((0, -1)\) on the circle.

2. The normal line is horizontal (and hence, the tangent line is vertical) when \(\sin t = 0\); that is, when \(t = 0, \pi, 2\pi\), corresponding to the points \((-1, 0)\) and \((0, 1)\) on the circle. These results should make intuitive sense.

The normal line is horizontal (and hence, the tangent line is vertical) when \(\sin t = 0\); that is, when \(t = 0, \pi, 2\pi\), corresponding to the points \((-1, 0)\) and \((0, 1)\) on the circle. These results should make intuitive sense.

2. The slope of the normal line at \(t = t_0\) is \(m = \frac{\sin t_0}{\cos t_0}\). This normal line goes through the point \((\cos t_0, \sin t_0)\), giving the line \(y = \frac{\sin t_0}{\cos t_0} (x - \cos t_0) + \sin t_0\) for \(\tan t_0 \neq 0\). It is an important fact to recognize that the normal lines to a circle pass through its center, as illustrated in Figure 9.30. Stated in another way, any line that passes through the center of a circle intersects the circle at right angles.
Example (PageIndex{3}): Tangent lines when \( \frac{dy}{dx} \) is not defined

Find the equation of the tangent line to the astroid \( x=\cos^3 t, \ y=\sin^3 t \) at \( t=0 \), shown in Figure 9.31.

**SOLUTION**

We start by finding \( x'(t) \) and \( y'(t) \):

\[
x'(t) = -3\sin t \cos^2 t, \quad y'(t) = 3\cos t \sin^2 t.
\]

Note that both of these are 0 at \( t=0 \); the curve is not smooth at \( t=0 \) forming a cusp on the graph. Evaluating \( \frac{dy}{dx} \) at this point returns the indeterminate form of "0/0".

We can, however, examine the slopes of tangent lines near \( t=0 \), and take the limit as \( t \to 0 \).

\[
\begin{align*}
\lim_{t \to 0} \frac{y'(t)}{x'(t)} &= \lim_{t \to 0} \frac{3\cos t \sin^2 t}{-3\sin t \cos^2 t} \\
&= \lim_{t \to 0} -\frac{\sin t}{\cos t} \\
&= 0.
\end{align*}
\]

We have accomplished something significant. When the derivative \( \frac{dy}{dx} \) returns an indeterminate form at \( t=t_0 \), we can define its value by setting it to be \( \lim_{t \to t_0} \frac{dy}{dx} \), if that limit exists. This allows us to find slopes of tangent lines at cusps, which can be very beneficial.
We found the slope of the tangent line at \((t=0)\) to be 0; therefore the tangent line is \((y=0)\), the \((x)\)-axis.

**Concavity**

We continue to analyze curves in the plane by considering their concavity; that is, we are interested in \(\left(\frac{d^2y}{dx^2}\right)\), "the second derivative of \((y)\) with respect to \((x)\)." To find this, we need to find the derivative of \(\left(\frac{dy}{dx}\right)\) with respect to \((x)\); that is, \(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)\) but recall that \(\left(\frac{dy}{dx}\right)\) is a function of \((t)\), not \((x)\), making this computation not straightforward.

To make the upcoming notation a bit simpler, let \(h(t) = \frac{dy}{dx}\). We want \(\frac{dh}{dx}\); that is, we want \(\frac{dh}{dt}\Bigg/\frac{dx}{dt}\). We again appeal to the Chain Rule. Note:

\[
\frac{dh}{dt} = \frac{dh}{dx}\cdot\frac{dx}{dt} \quad \Rightarrow \quad \frac{dh}{dx} = \frac{dh}{dt}\Bigg/\frac{dx}{dt}.
\]

In words, to find \(\frac{d^2y}{dx^2}\), we first take the derivative of \(\frac{dy}{dx}\) with respect to \((t)\), then divide by \((x)\prime (t)\). We restate this as a Key Idea.

**Key Idea 38 Finding \(\frac{d^2y}{dx^2}\) with Parametric Equations**

Let \((x=f(t))\) and \((y=g(t))\) be twice differentiable functions on an open interval \((I)\), where \((f)\prime (t)\neq 0\) on \((I)\). Then

\[
\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\Bigg/\frac{dx}{dt} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\Bigg/f\prime (t).
\]

Examples will help us understand this Key Idea.
Example 4: Concavity of Plane Curves

Let \(x=5t^2-6t+4\) and \(y=t^2+6t-1\) as in Example 9.3.1. Determine the \(t\)-intervals on which the graph is concave up/down.

**SOLUTION**

Concavity is determined by the second derivative of \(y\) with respect to \(x\), \(\frac{d^2y}{dx^2}\), so we compute that here following Key Idea 38.

In Example 9.3.1, we found \(\frac{dy}{dx} = \frac{2t+6}{10t-6}\) and \(f'(t) = 10t-6\). So:

\[
\begin{align*}
\frac{d^2y}{dx^2} &= \frac{d}{dt}\left[\frac{2t+6}{10t-6}\right]\Bigg/(10t-6) \\
&= -\frac{72}{(10t-6)^2}\Bigg/(10t-6) \\
&= -\frac{72}{(10t-6)^3} \\
&= -\frac{9}{(5t-3)^3}
\end{align*}
\]

The graph of the parametric functions is concave up when \(\frac{d^2y}{dx^2} > 0\) and concave down when \(\frac{d^2y}{dx^2} < 0\). We determine the intervals when the second derivative is greater/less than 0 by first finding when it is 0 or undefined.

As the numerator of \(-\frac{9}{(5t-3)^3}\) is never 0, \(\frac{d^2y}{dx^2} \neq 0\) for all \(t\). It is undefined when \((5t-3=0)\); that is, when \(t=3/5\). Following the work established in Section 3.4, we look at values of \(t\) greater/less than \(3/5\) on a number line:
Reviewing Example 9.3.1, we see that when \(t=3/5=0.6\), the graph of the parametric equations has a vertical tangent line. This point is also a point of inflection for the graph, illustrated in Figure 9.32.

Example \(\PageIndex{5}\): Concavity of Plane Curves

Find the points of inflection of the graph of the parametric equations \((x=\sqrt{t}), (y=\sin t)\), for \((0\leq t\leq 16)\).

SOLUTION

We need to compute \(\frac{dy}{dx}\) and \(\frac{d^2y}{dx^2}\).

\[
\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\cos t}{1/(2\sqrt{t})} = 2\sqrt{t}\cos t.
\]

\[
\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{x'(t)} = \frac{\cos t/\sqrt{t}-2\sqrt{t}\sin t}{1/(2\sqrt{t})} = 2\cos t-4t\sin t.
\]

The points of inflection are found by setting \(\frac{d^2y}{dx^2}=0\). This is not trivial, as equations that mix polynomials and trigonometric functions generally do not have "nice" solutions.

In Figure 9.33(a) we see a plot of the second derivative. It shows that it has zeros at approximately \((t=0.5, 3.5, 6.5, 9.5, 12.5)\) and \((16)\). These approximations are not very good, made only by looking at the graph. Newton's Method provides more accurate approximations. Accurate to 2 decimal places, we have:

\([t=0.65, 3.29, 6.36, 9.48, 12.61, 15.74]\)

The corresponding points have been plotted on the graph of the parametric equations in Figure 9.33(b). Note how most occur near the \((x)\)-axis, but not exactly on the axis.
Figure 9.33: In (a), a graph of \( \frac{d^2y}{dx^2} \), showing where it is approximately 0. In (b), graph of the parametric equations in Example 9.3.5 along with points of inflection.

**Arc Length**

We continue our study of the features of the graphs of parametric equations by computing their arc length. Recall in Section 7.4 we found the arc length of the graph of a function, from \( x=a \) to \( x=b \), to be

\[
\int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx.
\]

We can use this equation and convert it to the parametric equation context. Letting \( x=f(t) \) and \( y=g(t) \), we know that

\[
\left( \frac{dy}{dx} \right) = \frac{g'(t)}{f'(t)}.
\]

We will also be useful to calculate the differential of \( x \):

\[
dx = f'(t) \, dt \quad \Rightarrow \quad dt = \frac{1}{f'(t)} \cdot dx.
\]

\[
[L = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx.]
\]

We can use this equation and convert it to the parametric equation context. Letting \( x=f(t) \) and \( y=g(t) \), we know that

\[
\left( \frac{dy}{dx} \right) = \frac{g'(t)}{f'(t)}.
\]

It will also be useful to calculate the differential of \( x \):

\[
[dx = f'(t) \, dt \quad \Rightarrow \quad dt = \frac{1}{f'(t)} \, dx.]
\]
Starting with the arc length formula above, consider:

\[
\begin{align*}
L &= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2} \ dx \\
&= \int_a^b \sqrt{1+\frac{g'(t)^2}{f'(t)^2}} \ dx. \\
\text{Factor out the } f'(t)^2: & \quad \int_a^b \sqrt{f'(t)^2+g'(t)^2} \cdot \frac{1}{f'(t)} \ dx = dt \\
&= \int_{t_1}^{t_2} \sqrt{f'(t)^2+g'(t)^2} \ dt.
\end{align*}
\]

Note the new bounds (no longer "\((x)\)" bounds, but "\((t)\)" bounds). They are found by finding \(\langle t_1 \rangle\) and \(\langle t_2 \rangle\) such that \(a=f(t_1)\) and \(b=f(t_2)\). This formula is important, so we restate it as a theorem.

**Theorem 82: Arc Length of Parametric Curves**

Let \(x=f(t)\) and \(y=g(t)\) be parametric equations with \(f'(t)\) and \(g'(t)\) continuous on some open interval \(I\) containing \(\langle t_1 \rangle\) and \(\langle t_2 \rangle\) on which the graph traces itself only once. The arc length of the graph, from \(\langle t=t_1 \rangle\) to \(\langle t=t_2 \rangle\), is

\[L = \int_{t_1}^{t_2} \sqrt{f'(t)^2+g'(t)^2} \ dt.\]

As before, these integrals are often not easy to compute. We start with a simple example, then give another where we approximate the solution.

**Example (PageIndex{6}): Arc Length of a Circle**

Find the arc length of the circle parametrized by \((x=3\cos t), (y=3\sin t)\) on \([0,3\pi/2]\).

**Solution**

By direct application of Theorem 82, we have

\[
\begin{align*}
L &= \int_0^{3\pi/2} \sqrt{(-3\sin t)^2 +(3\cos t)^2} \ dt. \\
\text{Apply the Pythagorean Theorem:} & \quad \int_0^{3\pi/2} 3 \ dt \\
&= 3t \Big|_0^{3\pi/2} = 9\pi/2.
\end{align*}
\]

This should make sense; we know from geometry that the circumference of a circle with radius 3 is \(6\pi\); since we are finding the arc length of \((3/4)\) of a circle, the arc length is \((3/4)\cdot 6\pi = 9\pi/2\).

**Example (PageIndex{7}): Arc Length of a Parametric Curve**

The graph of the parametric equations \((x=t(t^2-1)), (y=t^2-1)\) crosses itself as shown in Figure 9.34, forming a "teardrop." Find the arc length of the teardrop.
SOLUTION

We can see by the parametrizations of \((x)\) and \((y)\) that when \((t=\pm1)\), \((x=0)\) and \((y=0)\). This means we'll integrate from \((t=-1)\) to \((t=1)\). Applying Theorem 82, we have

\[
\begin{align*}
L &= \int_{-1}^{1}\sqrt{(3t^2-1)^2+(2t)^2} \ dt \\
&= \int_{-1}^{1} \sqrt{9t^4-2t^2+1} \ dt.
\end{align*}
\]

Unfortunately, the integrand does not have an antiderivative expressible by elementary functions. We turn to numerical integration to approximate its value. Using 4 subintervals, Simpson's Rule approximates the value of the integral as \((2.65051)\). Using a computer, more subintervals are easy to employ, and \((n=20)\) gives a value of \((2.71559)\). Increasing \((n)\) shows that this value is stable and a good approximation of the actual value.

---

**Figure 9.34:** A graph of the parametric equations in Example 9.3.7, where the arc length of the teardrop is calculated.

---

**Surface Area of a Solid of Revolution**

Related to the formula for finding arc length is the formula for finding surface area. We can adapt the formula found in Key Idea 28 from Section 7.4 in a similar way as done to produce the formula for arc length done before.

**key idea 39 Surface Area of a Solid of Revolution**

Consider the graph of the parametric equations \((x=f(t))\) and \((y=g(t))\), where \((f'\) prime \()\) and \((g'\) prime \()\) are continuous on an open interval \((I)\) containing \((t_1)\) and \((t_2)\) on which the graph does not cross itself.

1. The surface area of the solid formed by revolving the graph about the \((x)\)-axis is (where \((g(t)\geq0)\) on \(([t_1,t_2])\)):

\[
\text{Surface Area} = 2\pi\int_{t_1}^{t_2} g(t)\sqrt{f'(t)^2+g'(t)^2} \ dt.
\]

2. The surface area of the solid formed by revolving the graph about the \((y)\)-axis is (where \((f(t)\geq0)\) on \(([t_1,t_2])\)):
\[
\text{Surface Area} = 2\pi \int_{t_1}^{t_2} f(t)\sqrt{f'(t)^2+g'(t)^2} \, dt.
\]

Example \(\PageIndex{8}\): Surface Area of a Solid of Revolution

Consider the teardrop shape formed by the parametric equations \((x=t(t^2-1)), (y=t^2-1)\) as seen in Example 9.3.7. Find the surface area if this shape is rotated about the \((x)\)-axis, as shown in Figure 9.3.8.

![Figure 9.35: Rotating a teardrop shape about the x-axis in Example 9.3.8](image)

Solution

The teardrop shape is formed between \((t=-1)\) and \((t=1)\). Using Key Idea 39, we see we need for \((g(t)\geq 0)\) on \([-1,1]\), and this is not the case. To fix this, we simplify replace \((g(t))\) with \((-g(t))\), which flips the whole graph about the \((x)\)-axis (and does not change the surface area of the resulting solid). The surface area is:

\[
\begin{align*}
\text{Area} \ S &= 2\pi \int_{-1}^{1} (1-t^2)\sqrt{(3t^2-1)^2+(2t)^2} \, dt \\
&= 2\pi \int_{-1}^{1} (1-t^2)\sqrt{9t^4-2t^2+1} \, dt.
\end{align*}
\]

Once again we arrive at an integral that we cannot compute in terms of elementary functions. Using Simpson's Rule with \((n=20)\), we find the area to be \((S=9.44)\). Using larger values of \((n)\) shows this is accurate to 2 places after the decimal.

After defining a new way of creating curves in the plane, in this section we have applied calculus techniques to the parametric equation defining these curves to study their properties. In the next section, we define another way of forming curves in the plane. To do so, we create a new coordinate system, called \textit{polar coordinates}, that identifies points in the plane in a manner different than from measuring distances from the \((y)\)- and \((x)\)-axes.
Contributors

- Gregory Hartman (Virginia Military Institute). Contributions were made by Troy Siemers and Dimplekumar Chalishajar of VMI and Brian Heinold of Mount Saint Mary’s University. This content is copyrighted by a Creative Commons Attribution-Noncommercial (BY-NC) License. http://www.apexcalculus.com/